The envelope theorem.

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1. *Variables and parameters*

Most economic functions depend on both *endogenous variables*, which we usually just call 'variables', and *exogenous variables* which are typically called 'parameters'. Endogenous means '*having an internal cause*' and endogenous variables are the variables whose values are chosen within the model being studied.† Exogenous means '*having an external cause*', and the values of exogenous variables are typically determined outside that model.‡

As an example, consider the profit function of a monopolistic firm,

$$\Pi = f(p; \alpha, \beta, \gamma) = -\alpha p^2 + \beta p - \gamma, \tag{1}$$

that depends on the price p that the firm sets for its product, and the quantities α, β and γ that are determined by the market in which the firm operates. From the firm's perspective,

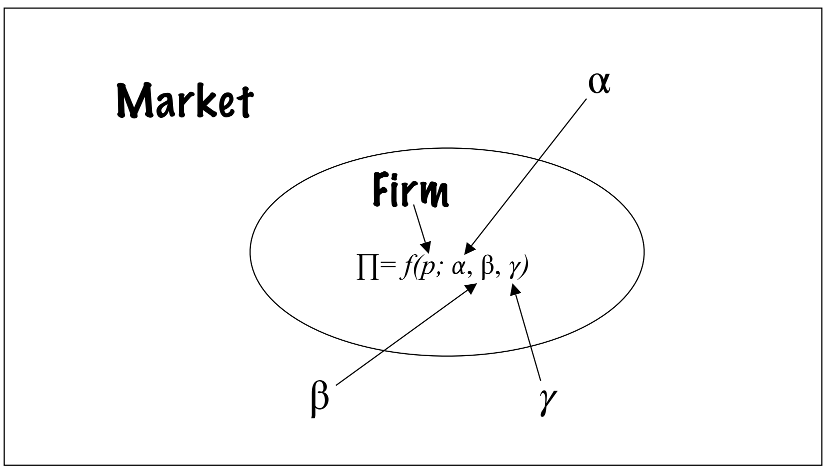

Figure 1: The firm's profit model in the bigger model of the market.

the only variable whose value it chooses is the price p so this is the only (endogenous) variable in the firm's profit function. The values of α, β and γ are fixed outside the firm's profit model (the oval in Figure 1), so they are the exogenous variables, or parameters, in this case. To make this distinction clear in the notation, people often use a semicolon (;) to separate the variables from the parameters, as I did in (1).

The distinction between variables and parameters becomes very important in the context of optimization. For example, the firm above can only change its profit by changing its price because it has no control over the parameters—the firm treats the parameters as *constants* in this context. Therefore, to maximize its profit, the firm treats the profit function as a function of the single variable p and follows the usual steps.

(i) Differentiate Π with respect to p: $\dfrac{d\Pi}{dp} = -2\alpha p + \beta$.

†Such variables are also sometimes called 'choice variables'.
‡But inside a bigger model.

(ii) Find the critical price where this derivative is equal to 0. That is, solve the equation $d\Pi/dp = 0$ for the variable p:

$$\frac{d\Pi}{dp} = 0 \implies -2\alpha p + \beta = 0 \implies p^* = \frac{\beta}{2\alpha}.$$

(iii) Verify that the critical profit is indeed a maximum

$$\frac{d^2\Pi}{dp^2} = -2\alpha.$$

So if $\alpha > 0$ (which means that $-2\alpha < 0$) then

$$\Pi^* = -\alpha\,(p^*)^2 + \beta p^* - \gamma = \frac{\beta^2}{4\alpha} - \gamma$$

is the firm's maximum profit.

This simple example illustrates two important aspects of the roles that variables and parameters play in the context of optimization.

- During the optimization process itself (the differentiation, solving the equation(s), etc.), only the (endogenous) variables are treated as variables. I.e., we only differentiate with respect to the variables, and we only treat the variables as unknowns in the resulting equation(s). *The parameters are treated as constants.*
- The critical values (p^* and Π^* in the example above) that we find are themselves *functions of the parameters.*

The second feature above, raises the question that this note is concerned with:

How do changes in the value(s) of the parameter(s) affect the critical value of the function?

In the example we have been following, this question is fairly easy to answer because we have an explicit expression

$$\Pi^* = \Pi^*(\alpha, \beta, \gamma) = \frac{\beta^2}{4\alpha} - \gamma$$

for the critical value of profit as a function of the three parameters, so we can compute the partial derivatives of Π with respect to each of the variables and use linear approximation to estimate the change in maximum profit that would occur if one or more of the variables changed a little.

To illustrate, suppose that $\alpha = 0.4$, $\beta = 60$ and $\gamma = 500$, then

$$p^* = \frac{60}{0.8} = 75 \text{ and } \Pi^* = \frac{3600}{1.6} - 500 = 1750.$$

Now,

$$\frac{\partial \Pi^*}{\partial \alpha} = -\frac{\beta^2}{4\alpha^2}$$

and therefore

$$\left.\frac{\partial \Pi^*}{\partial \alpha}\right|_{\substack{\alpha=0.4\\ \beta=60\\ \gamma=500}} = -\frac{3600}{0.64} = -5625,$$

so if the parameter α increases to 0.42 (i.e., $\Delta\alpha = 0.02$), due to market forces, then the firm's maximum profit would change by

$$\Delta\Pi^* \approx \left.\frac{\partial\Pi^*}{\partial\alpha}\right|_{\substack{\alpha=0.4\\ \beta=60\\ \gamma=500}} \cdot \Delta\alpha = (-5625)\cdot(0.02) = -112.5.$$

That is the firm's (maximum) profit would decrease by about 112.5.

Of course in this case, we can simply calculate the change in profit directly:

$$\Delta\Pi^* = \Pi^*(0.42, 60, 500) - \Pi^*(0.4, 60, 500) \approx 1642.86 - 1750 = -107.14,$$

which shows that the estimated change we found was reasonably close.

In general, however, the relationship between the critical value of the function being optimized and the parameters involved in its definition may not be quite so simple to write down. This is where the *Envelope theorem* is useful.

2. *The envelope theorem*

Suppose that we have a function $F(x, y, z; \alpha, \beta)$ which depends on the three variables x, y and z and the two parameters α and β.[§] To find the critical value(s) of this function, we find the critical point(s) (x^*, y^*, z^*) that satisfy first order conditions

$$\left.\begin{array}{rcl} F_x(x, y, z; \alpha, \beta) & = & 0 \\ F_y(x, y, z; \alpha, \beta) & = & 0 \\ F_z(x, y, z; \alpha, \beta) & = & 0 \end{array}\right\} \tag{2}$$

and the value of the function F at this point

$$F^* = F(x^*, y^*, z^*; \alpha, \beta)$$

is the corresponding critical value.

As in the profit maximizing example in the previous section, the parameters α and β are treated as constants throughout the differentiation and equation-solving process. The key observation is the following.

> *The coefficients that appear in the system of equations* (2) *depend on the values of the parameters α and β. Therefore the solution of this system, (x^*, y^*, z^*), also depends on these values, as does the critical value F^*.*

We can express this symbolically by writing

$$x^* = x^*(\alpha, \beta),\ y^* = y^*(\alpha, \beta),\ z^* = z^*(\alpha, \beta)$$

and most importantly here

$$F^* = F(x^*, y^*, z^*; \alpha, \beta) = F^*(\alpha, \beta). \tag{3}$$

While the function $F(x, y, z; \alpha, \beta)$ may be given explicitly, it can be complicated to find the explicit functional relationship $F^* = F^*(\alpha, \beta)$. The envelope theorem allows us to

[§]There is nothing special about the number of variables vs. the number of parameters, as long as there is at least one of each.

sidestep this difficulty and compute the derivatives of F^* with respect to the parameters α and β in spite of this.

Suppose that we want to find $\partial F^*/\partial\alpha$. To do this, we differentiate $F^* = F(x^*, y^*, z^*; \alpha, \beta)$ with respect to α, using the *chain rule*:[¶]

$$\begin{aligned}\frac{\partial F^*}{d\alpha} &= \frac{\partial}{\partial\alpha}\big[F(x^*, y^*, z^*; \alpha, \beta)\big] \\ &= F_x(x^*, y^*, z^*; \alpha, \beta)\frac{\partial x^*}{\partial\alpha} + F_y(x^*, y^*, z^*; \alpha, \beta)\frac{\partial y^*}{\partial\alpha} + F_z(x^*, y^*, z^*; \alpha, \beta)\frac{\partial z^*}{\partial\alpha} \\ &\qquad + F_\alpha(x^*, y^*, z^*; \alpha, \beta)\frac{d\alpha}{d\alpha} + F_\beta(x^*, y^*, z^*; \alpha, \beta)\frac{d\beta}{d\alpha}.\end{aligned} \tag{4}$$

Now we use the fact that (x^*, y^*, z^*) is a critical point of the function $F(x, y, z; \alpha, \beta)$, which means that

$$F_x(x^*, y^*, z^*; \alpha, \beta) = 0, \quad F_y(x^*, y^*, z^*; \alpha, \beta) = 0 \;\text{ and }\; F_z(x^*, y^*, z^*; \alpha, \beta) = 0,$$

so the three terms in the middle row of (4) all vanish! Additionally, we have $d\beta/d\alpha = 0$ (which follows from the tacit assumption that α and β are independent of each other) and $d\alpha/d\alpha = 1$, so in conclusion we see that

$$\frac{\partial F^*}{d\alpha} = F_\alpha(x^*, y^*, z^*; \alpha, \beta). \tag{5}$$

It is important to note that the envelope theorem doesn't require that F^* be an optimal value, merely that it be a *critical* value. I.e., the envelope theorem only requires that the first order conditions hold at the point $(x^*(\alpha, \beta), y^*(\alpha, \beta), z^*(\alpha, \beta))$.

3. *Examples*

Example 1. First I'll take another look at the profit-maximization example of §1. The envelope theorem says that

$$\frac{\partial\Pi^*}{\partial\alpha} = \Pi_\alpha(p^*; \alpha, \beta, \gamma) = \frac{\partial}{\partial\alpha}\big[-\alpha(p^*)^2 + \beta p^* - \gamma\big] = -(p^*)^2.$$

When $\alpha = 0.4$, $\beta = 60$ and $\gamma = 500$, we found that $p^* = 75$, and therefore

$$\left.\frac{\partial\Pi^*}{\partial\alpha}\right|_{\substack{\alpha=0.4\\ \beta=60\\ \gamma=500}} = -75^2 = -5625,$$

which is exactly what we found before.

Example 2. A monopolistic firm sells one product in two markets, A and B. The daily demand equations for the firm's product in these markets are given by

$$Q_A = 100 - 0.4P_A \qquad \text{and} \qquad Q_B = 120 - 0.5P_B,$$

where Q_A and Q_B are the daily demands and P_A and P_B are the prices for the firm's product in each market, respectively. The firm's constant marginal cost is $\alpha = \$40$ and the

[¶]See section 17.5 in the textbook.

its daily fixed cost is $\beta = \$2500$. The parameters α and β cannot be changed by the firm, depending as they do on things like the costs of the firm's inputs, labor costs and so forth.[‖]

The firm wants to find the prices that it should set for each market to maximize its daily profit, and the first step is to find its daily profit as a function of the prices. First, the firm's daily revenue is

$$R = P_A Q_A + P_B Q_B = P_A(100 - 0.4P_A) + P_B(120 - 0.5P_B),$$

and its daily cost is

$$C = \alpha(Q_A + Q_B) + \beta = 40(100 - 0.4P_A + 120 - 0.5P_B) + 2500.$$

So the firm's profit function is

$$\begin{aligned} \Pi(P_A, P_B; \alpha, \beta) &= R - C = P_A Q_A + P_B Q_B - \alpha(Q_A + Q_B) - \beta \qquad (6) \\ &= P_A(100 - 0.4P_A) + P_B(120 - 0.5P_B) - (40(100 - 0.4P_A + 120 - 0.5P_B) + 2500) \\ &= -0.4P_A^2 - 0.5P_B^2 + 116P_A + 140P_B - 11300. \end{aligned}$$

To find the critical prices, we solve the first order equations

$$\left.\begin{array}{rcl} \Pi_{P_A} &=& 0 \\ \Pi_{P_B} &=& 0 \end{array}\right\} \implies \left.\begin{array}{rcl} -0.8P_A + 116 &=& 0 \\ -P_B + 140 &=& 0 \end{array}\right\} \implies \begin{array}{rcl} P_A^* &=& 145 \\ P_B^* &=& 140 \end{array}$$

The corresponding critical output (demand) levels are

$$Q_A^* = 100 - 0.4P_A^* = 42 \quad \text{and} \quad Q_B^* = 120 - 0.5P_B^* = 50,$$

and the critical value of the profit function is

$$\Pi^* = P_A^* Q_A^* + P_B^* Q_B^* - 40(Q_A^* + Q_B^*) - 2500 = 6910.$$

Finally, the discriminant for the profit function is positive,

$$D = \Pi_{P_A P_A}\Pi_{P_B P_B} - \Pi_{P_A P_B}^2 = (-0.8)(-1) - 0^2 = 0.8 > 0,$$

which together with the fact that $\Pi_{P_A P_A} = -0.8 < 0$, show that Π^* is the firm's maximum profit, by the second derivative test.

Now, suppose that the firm's marginal cost *increases* from $\alpha = 40$ to $\alpha = 40.5$ because the price of one of their inputs has increased. By how much will the firm's maximum daily profit change? One way to answer this question is to simply redo the optimization problem, which is not difficult in this case. But instead, I will use the envelope theorem to find $\partial\Pi^*/\partial\alpha$, and then use *linear approximation* to estimate the change in the profit.

From the envelope theorem and studying the first line in Equation (6), we can see that

$$\frac{\partial \Pi^*}{\partial \alpha} = \Pi_\alpha(P_A^*, P_B^*; \alpha, \beta) = -(Q_A^* + Q_B^*) = -92.$$

Therefore the change in the firm's profit can be estimated as

$$\Delta\Pi^* \approx \frac{\partial \Pi^*}{\partial \alpha} \cdot \Delta\alpha = (-92)(0.5) = -46,$$

so that the firm's profit will decrease by about \$46 a day.

[‖] The firm can *influence* the values of these parameters in the long run. For example it can change suppliers (to lower the price of inputs) or it can change its labor costs by bargaining with its employees, etc. But it doesn't do this on a daily basis, so for the purposes of the daily profit maximization problem, we consider the marginal and fixed costs to be *exogenous* variables.