Study Guide 9

Solutions

1. A monopolistic firm sells one product in two markets, A and B. The daily demand equations for the firm's product in these markets are given by

$$Q_A = 100 - 0.4P_A \qquad \text{and} \qquad Q_B = 120 - 0.5P_B,$$

where Q_A and Q_B are the demands and P_A and P_B are the prices for the firm's product in markets A and B, respectively. The firm's constant marginal cost is \$40 and the its daily fixed cost is \$2500.

a. Find the prices that the firm should charge in each market to maximize its daily profit. Use the second derivative test to verify that the prices you found yield the *absolute* maximum profit.

Solution: *The firm's total daily output is* $Q_A + Q_B$*, so the firm's daily cost is*

$$C = 40(Q_A + Q_B) + 2500.$$

The firm's revenue from markets A and B is $R_A = P_A Q_A$ *and* $R_B = P_B Q_B$*. The firm's profit function is*

$$\begin{aligned}
\Pi &= P_A Q_A + P_B Q_B - C \\
&= P_A(100 - 0.4P_A) + P_B(120 - 0.5P_B) - [40(100 - 0.4P_A + 120 - 0.5P_B) + 2500] \\
&= -0.4P_A^2 - 0.5P_B^2 + 116P_A + 140P_B - 11300.
\end{aligned}$$

The first order conditions for an optimum value give

$$\begin{aligned}
\Pi_{P_A} &= 0 \implies -0.8P_A + 116 = 0 \implies \boxed{P_A^* = 145} \\
\Pi_{P_B} &= 0 \implies -P_B + 140 = 0 \implies \boxed{P_B^* = 140}
\end{aligned}$$

The second order conditions for a maximum are

$$\Pi_{P_A P_A}(P_A^*, P_B^*) \cdot \Pi_{P_B P_B}(P_A^*, P_B^*) - (\Pi_{P_A P_B}(P_A^*, P_B^*))^2 > 0 \;\; \textit{and} \;\; \Pi_{P_A P_A}(P_A^*, P_B^*) < 0.$$

In this case we have $\Pi_{P_A P_A} = -0.8 < 0$ *and*

$$\Pi_{P_A P_A}(P_A^*, P_B^*) \cdot \Pi_{P_B P_B}(P_A^*, P_B^*) - (\Pi_{P_A P_B}(P_A^*, P_B^*))^2 = 0.8 > 0$$

for all (P_A, P_B)*, so that the second order conditions are satisfied, and because the conditions hold everywhere, the critical value of profit*

$$\Pi^* = \Pi(P_A^*, P_B^*) = 6910$$

is the absolute maximum daily profit.

To summarize: the firm's profit is maximized when $P_A^* = 145$ *and* $P_B^* = 140$*, at which point the daily demands are*

$$Q_A^* = 100 - 0.4P_A^* = 42 \quad \textit{and} \quad Q_B^* = 120 - 0.5P_B^* = 50$$

and the maximum profit is $\Pi^* = 6910$.

b. Use the *envelope theorem* (and linear approximation) to estimate the change in the Firm's max profit if the marginal cost of their product increases to \$40.75.

Solution: *If we denote the marginal cost by μ, then the profit function can be written as*

$$\Pi = P_A Q_A + P_B Q_B - \mu \cdot \overbrace{(Q_A + Q_B)}^{\textit{total daily ouptut}} - 2500.$$

According to the ***envelope theorem***

$$\frac{d\Pi^*}{d\mu} = \frac{d\Pi}{d\mu}\Bigg|_{\substack{P_A = P_A^* \\ P_B = P_B^*}} = -(Q_A + Q_B)\Bigg|_{\substack{P_A = P_A^* \\ P_B = P_B^*}} = -(Q_A^* + Q_B^*) = -92.$$

Now, using ***linear approximation****, we find that*

$$\Delta\Pi^* \approx \frac{d\Pi^*}{d\mu} \cdot \Delta\mu = -92 \cdot (0.75) = -69.$$

I.e., if the firm's marginal cost increases by \$0.75, then the max daily profit will decrease by about \$69.00.

2. Jack's (gustatory) utility function is

$$U(x, y, z) = 5 \ln x + 7 \ln y + 18 \ln z,$$

where x is the number of fast-food meals Jack consumes in a month; y is the number of 'diner' meals he consumes in a month; and z is the number of 'fancy restaurant' meals he consumes in a month.

The average price of a fast-food meal is $p_x = \$4.00$; the average price of a 'diner' meal is $p_y = \$8.00$; and the average price of a 'fancy restaurant' meal is $p_z = \$30.00$.

a. How many meals of each type should Jack consumer per month to maximize his utility, if his monthly budget for these meals is $\beta = \$1200.00$?

Solution: *The objective function is the utility $U(x, y, z) = 5 \ln x + 7 \ln y + 18 \ln z$, and the constraint is the budget (or income) constraint we obtain from the prices and the budget:*

$$xp_x + yp_y + zp_z = \beta \implies 4x + 8y + 30z = 1200.$$

Lagrangian: $F(x, y, z, \lambda) = 5 \ln x + 7 \ln y + 18 \ln z - \lambda(4x + 8y + 30z - 1200)$.

'Structural' equations:

$$\begin{aligned} F_x &= \frac{5}{x} - 4\lambda &= 0 \\ F_y &= \frac{7}{y} - 8\lambda &= 0 \\ F_z &= \frac{18}{z} - 30\lambda &= 0. \end{aligned}$$

Solving these equations for λ gives the triple equation

$$\lambda = \frac{5}{4x} = \frac{7}{8y} = \frac{3}{5z}.$$

Comparing the x-term and the y-term and clearing denominators gives

$$\frac{5}{4x} = \frac{7}{8y} \implies 40y = 28x \implies y = \frac{7x}{10}.$$

Comparing the x-term and the z-term and clearing denominators gives

$$\frac{5}{4x} = \frac{3}{5z} \implies 25z = 12x \implies z = \frac{12x}{25}.$$

Substituting the expressions for y and z that we found into the budget constraint ($F_\lambda = 0$) gives

$$4x + 8\left(\frac{7x}{10}\right) + 30\left(\frac{12x}{25}\right) = 1200 \implies 1200x = 60000 \implies \boxed{x^* = 50,\ y^* = 35,\ z^* = 24.}$$

Thus, Jack maximizes his utility by consuming 50 fast food meals, 35 diner meals and 24 'fancy' restaurant meals in a month, resulting in a max utility of

$$U^* = U(x^*, y^*, z^*) = U(50, 35, 24) \approx 101.652.$$

b. By approximately how much will Jack's utility increase if his budget increases by \$50.00? Explain your answer.

Solution: *Since the utility function and the prices of meals are not changing, the maximum utility, U^*, is a function of the budget, β. I.e., increasing the budget increases U^* and decreasing the budget decreases U^*.*

Now, observe that at the critical point (x^*, y^*, z^*)

$$F^* = F(x^*, y^*, z^*, \lambda; \beta) = U(x^*, y^*, z^*) - \lambda\overbrace{(4x^* + 8y^* + 30z^* - 1200)}^{=0 \quad \textit{because } F_\lambda = 0} = U(x^*, y^*, z^*) = U^*,$$

which means that

$$\frac{dU^*}{d\beta} = \frac{dF^*}{d\beta}.$$

*Next, the **envelope theorem** applied to the Lagrangian function $F(x, y, z, \lambda; \beta)$ tells us that*

$$\frac{dU^*}{d\beta} = \frac{dF^*}{d\beta} = \left.\frac{dF}{d\beta}\right|_{\substack{x=x^*\\ y=y^*\\ z=z^*\\ \lambda=\lambda^*}} = \lambda^*,$$

where λ^* is the critical value of the multiplier λ. In this case,

$$\lambda^* = \frac{5}{4x^*} = \frac{5}{200} = 0.025.$$

Finally, we use linear approximation:

$$\Delta U^* \approx \frac{dU^*}{d\beta} \cdot \Delta\beta = \lambda^* \cdot \Delta\beta = 0.025 \cdot 50 = 1.25.$$

In other words, if Jack's food budget increases by \$50.00, then his max utility will increase by approximately 1.25.

3. A firm's productions function is given by

$$Q = 10K^{0.4}L^{0.7},$$

where Q is the firm's annual output, K is the annual capital input, and L is the annual labor input. The cost per unit of capital is \$1000, and the cost per unit of labor is \$4000.

a. Find the levels of labor and capital inputs that **minimize** the cost of producing an output of $Q = 20,000$ units. What is the minimum cost?

Solution: *The firm's cost is the cost of using K units of capital and L units of labor, i.e., the **objective function** here is*

$$C(K, L) = 1000K + 4000L.$$

*The **constraint** in this case is the output target*

$$Q = 20,000 \implies 10K^{0.4}L^{0.7} = 20000,$$

so the Lagrangian is

$$F(K, L, \lambda) = 1000K + 4000L - \lambda(10K^{0.4}L^{0.7} - 20000).$$

The first-order equations are

$$\begin{aligned} F_K = 0 &\implies & 1000 - 4\lambda K^{-0.6}L^{0.7} = 0 \\ F_L = 0 &\implies & 4000 - 7\lambda K^{0.4}L^{-0.3} = 0 \\ F_\lambda = 0 &\implies & -(10K^{0.4}L^{0.7} - 20000) = 0 \end{aligned}$$

Solving the first two equations for λ gives

$$\lambda = \frac{1000}{4K^{-0.6}L^{0.7}} = \frac{4000}{7K^{0.4}L^{-0.3}} \implies 250\frac{K^{0.6}}{L^{0.7}} = \frac{4000}{7} \cdot \frac{L^{0.3}}{K^{0.4}}$$

Next, clear denominators in the equation on the right and solve for K in terms of L:

$$1750K = 4000L \implies K = \frac{16}{7}L.$$

Finally, substitute for K in the equation $F_\lambda = 0$ (the constraint), and solve for L:

$$10K^{0.4}L^{0.7} = 20000 \implies \overbrace{10\left(\frac{16}{7}L\right)^{0.4} \cdot L^{0.7} = 20000}^{(*)} \implies L^{1.1} = \frac{2000}{(16/7)^{0.4}}$$

$$\Longrightarrow L = \left(\frac{2000}{(16/7)^{0.4}}\right)^{1/1.1} \quad \Longrightarrow L^* \approx 741.964$$

Conclusion: *Cost is minimized when* $L^* \approx 741.964$ *and* $K^* = \frac{16}{7}L^* \approx 1695.918$. *The minimum cost is*

$$C^* = \$1000K^* + \$4000L^* \approx \$1,695,918 + \$2,967,856 = \$4,663,774$$

b. Find the levels of labor and capital inputs that **minimize** the cost of producing an output of $Q = q$ units and find the minimum cost. Express your answer in terms of q.

Solution: *There is no need to start over from the beginning. The only difference between this and* **a.** *is that the target output changes from* 20000 *to* q. *This means that we can skip directly to the equation where* 20000 *makes its first appearance, namely the equation marked with a (*) above, and replace the* 20000 *that appears there by* q*:*

$$10\left(\frac{16}{7}L\right)^{0.4} \cdot L^{0.7} = q.$$

Now we continue as before to solve for L*, then* K *and finally,* C*. First* L*:*

$$10\left(\frac{16}{7}L\right)^{0.4} \cdot L^{0.7} = q \Longrightarrow L^{1.1} = \frac{q}{10(16/7)^{0.4}} \Longrightarrow L^*(q) = \frac{q^{10/11}}{10^{10/11}(16/7)^{4/11}} = \alpha \cdot q^{10/11},$$

where

$$\alpha = \left(\frac{7}{16}\right)^{4/11} 10^{-10/11} \approx 0.0913 \quad \left(\text{and } \frac{10}{11} = \frac{1}{1.1} \text{ and } \frac{4}{11} = \frac{4}{10} \cdot \frac{10}{11}\right)$$

Next,

$$K^*(q) = \frac{16}{7}L^*(q) = \beta \cdot q^{10/11},$$

where

$$\beta = \frac{16}{7}\alpha \approx 0.2086.$$

Finally, the (minimum) cost of producing q *units is*

$$C^*(q) = 1000K^*(q) + 4000L^*(q) = (1000\beta + 4000\alpha)q^{10/11} \approx 573.73q^{10/11}.$$

4. The production function for ACME Widgets is

$$Q = 2k^2 + kl + 5l^2,$$

where k and l are the numbers of units of capital and labor input, respectively, and Q is their output, *measured in 1000s of widgets*. The price per unit of capital input is $p_k = \$1000$ and the price per unit of labor input is $p_l = \$2500$.

a. How many units of capital and labor input should ACME use to ***minimize the cost*** of producing 65000 widgets? What is the *average cost per widget*?

Solution: *Since Q is measured in 1000s of widgets, the constraint here is*

$$2k^2 + kl + 5l^2 = 65$$

and the Lagrangian for this problem is

$$F(k, l, \lambda) = 1000k + 2500l - \lambda(2k^2 + kl + 5l^2 - 65).$$

The first order conditions are

$$\begin{aligned} F_k = 0 &\implies \quad 1000 - \lambda(4k + l) = 0 \\ F_l = 0 &\implies \quad 2500 - \lambda(k + 10l) = 0 \\ F_\lambda = 0 &\implies \quad 2k^2 + kl + 5l^2 = 65 \end{aligned}$$

Solving the first two equations for λ *gives*

$$\lambda = \frac{1000}{4k + l} = \frac{2500}{k + 10l}.$$

Dividing by 500 and clearing denominators on the right we find that

$$2k + 20l = 20k + 5l \implies 15l = 18k \implies l = \frac{6k}{5}.$$

Substituting for l *in the constraint we have*

$$2k^2 + k \cdot \left(\frac{6k}{5}\right) + 5\left(\frac{6k}{5}\right)^2 = 65 \implies \frac{52k^2}{5} = 65 \implies k^2 = 6.25.$$

Since capital input must be positive, the cost-minimizing levels of capital and labor input for producing 65000 widgets are $k^* = 2.5$ *and* $l^* = 3$*. It follows that the (minimum) cost of producing 65000 widgets is* $c^* = 1000 \cdot 2.5 + 2500 \cdot 3 = \$10000$*, and the average cost per widget is*

$$\bar{c}^* = \frac{10000}{65000} \approx \$0.154.$$

b. By approximately how much will ACME's cost rise if they raise their output from 65000 widgets to 65500 widgets? Justify your answer.

Solution: *Linear approximation tells us that* $\Delta c^* \approx \dfrac{dc^*}{dQ} \cdot \Delta Q$*, and by the envelope theorem, we have* $dc^*/dQ = \lambda^*$*, the critical value of the multiplier. In this problem we have*

$$\lambda^* = \frac{1000}{4k^* + l^*} = \frac{1000}{13}.$$

Also, if output increases by 500 widgets, then $\Delta Q = 0.5$*. Thus we have*

$$\Delta c^* \approx \frac{dc^*}{dQ} \cdot \Delta Q = \lambda^* \cdot \Delta Q = \frac{500}{13} \approx 38.46.$$

c. By approximately how much will ACME's minimum cost increase (from part a.) if the cost per unit of capital increases to \$1100? Use the *envelope theorem* and linear approximation.

Solution: *First observe that similarly to problem* **3b.**, *if we denote the price of capital by p_k, then*

$$\frac{dc^*}{dp_k} = \frac{dF^*}{dp_k},$$

where $F^* = F(k^*, l^*, \lambda^*; p_k)$ *and*

$$F = F(k, l, \lambda; p_k) = p_k k + 2500l - \lambda(2k^2 + kl + 5l^2 - 65).$$

Now use the envelope theorem to find dF^*/dp_k*:*

$$\frac{dc^*}{dp_k} = \frac{dF^*}{dp_k} = \frac{dF}{dp_k}\bigg|_{\substack{k=k^*\\ l=l^*\\ \lambda=\lambda^*}} = k\bigg|_{\substack{k=k^*\\ l=l^*\\ \lambda=\lambda^*}} = k^* = 2.5.$$

Finally, use this and linear approximation to find that

$$\Delta c^* \approx \frac{dc^*}{dp_k} \cdot \Delta p_k = 2.5 \cdot 100 = 250.$$

I.e., if the price per unit of capital increases by \$100*, then the cost will increase by about* \$250.

5. The annual output for a luxury hotel chain is given by $Q = 30K^{2/5}L^{1/2}R^{1/4}$, where K, L and R are the capital, labor and real estate inputs, all measured in \$1,000,000 s, and Q is the average number of rooms rented per day.

The hotel chain's annual budget is $B = \$69$ million.

a. How should they allocate this budget to the three inputs in order to *maximize* their annual output? What is the maximum output?

Solution: *We want to maximize the output,* $Q = 30K^{2/5}L^{1/2}R^{1/4}$*, subject to the budget constraint* $K + L + R = 69$*, since the inputs are all being measured in millions of dollars. The Lagrangian for this problem is*

$$F(K, L, R, \lambda) = 30K^{2/5}L^{1/2}R^{1/4} - \lambda(K + L + R - 69),$$

and the first order conditions are

$$\begin{aligned}
F_K = 0 \quad &\Longrightarrow \quad 12K^{-3/5}L^{1/2}R^{1/4} = \lambda,\\
F_L = 0 \quad &\Longrightarrow \quad 15K^{2/5}L^{-1/2}R^{1/4} = \lambda,\\
F_R = 0 \quad &\Longrightarrow \quad 7.5K^{2/5}L^{1/2}R^{-3/4} = \lambda,\\
F_\lambda = 0 \quad &\Longrightarrow \quad K + L + R = 69.
\end{aligned}$$

The first two equations imply that

$$12K^{-3/5}L^{1/2}R^{1/4} = 15K^{2/5}L^{-1/2}R^{1/4} \implies \frac{12L^{1/2}}{K^{3/5}} = \frac{15K^{2/5}}{L^{1/2}},$$

after canceling the common factor of $R^{1/4}$. *Clearing denominators gives*

$$12L = 15K \implies \boxed{L = 1.25K}.$$

Likewise, comparing the first and third equations implies that

$$12K^{-3/5}L^{1/2}R^{1/4} = 7.5K^{2/5}L^{1/2}R^{-3/4} \implies \frac{12R^{1/4}}{K^{3/5}} = \frac{7.5K^{2/5}}{R^{3/4}},$$

and clearing denominators gives

$$12R = 7.5K \implies \boxed{R = 0.625K}.$$

Substituting for R *and* L *in the fourth equation (the constraint) gives*

$$K + 1.25K + 0.625K = 69 \implies 2.875K = 69.$$

Thus, the critical values for the inputs are

$$K^* = \frac{69}{2.875} = 24, \quad L^* = 1.25K^* = 30 \ \textit{ and } \ R^* = 0.625K^* = 15,$$

and the hotel chain's maximum output is

$$Q^* = Q(24, 30, 15) \approx 1152.894.$$

b. What is the critical value of the multiplier when output is maximized?

Solution: *When output is maximized, the critical value of* λ *is*

$$\lambda^* = 12(K^*)^{-3/5}(L^*)^{1/2}(R^*)^{1/4} \approx 19.215.$$

c. Use your answer to **b.** to compute the *approximate* change in the firm's maximum output if their annual budget increases by \$500,000? Explain your answer.

Solution: *As in problem* **3b.**, *it follows from the envelope theorem that*

$$\frac{dQ^*}{dB} = \lambda^*,$$

where B *is the budget. It follows that*

$$\Delta Q^* \approx \lambda^* \cdot \Delta B \approx 19.215 \cdot 0.5 \approx 9.607,$$

since we measure the budget in the same units (millions of dollars) as the inputs, so an increase of \$500,000, means that $\Delta B = 0.5$.