## Solutions

1. Find the critical points of the functions below.
a. $f(x, y)=3 x^{2}-12 x y+19 y^{2}-2 x-4 y+5$. First order conditions:

$$
\left.\begin{array}{r}
f_{x}=6 x-12 y-2=0 \\
f_{y}=-12 x+38 y-4=0
\end{array}\right\}
$$

Now, $f_{x}=0 \Longrightarrow 6 x=12 y+2$, so $12 x=24 y+4$. Plugging this into the second equation gives

$$
\Longrightarrow-(24 y+4)+38 y-4=0 \Longrightarrow 14 y-8=0 \Longrightarrow y_{0}=\frac{4}{7} \Longrightarrow x_{0}=\frac{31}{21} .
$$

So there is one critical point, $\left(\frac{31}{21}, \frac{4}{7}\right)$.
b. $g(s, t)=s^{3}+3 t^{2}+12 s t+2$. First order conditions:

$$
\left.\begin{array}{r}
g_{s}=3 s^{2}+12 t=0 \\
g_{t}=6 t+12 s=0
\end{array}\right\}
$$

Now, $g_{t}=0 \Longrightarrow t=-2 s$, and plugging this into the first equation gives

$$
3 s^{2}-24 s=0 \Longrightarrow 3 s(s-8)=0 \Longrightarrow \text { two solutions: } s_{1}=0 \text { and } s_{2}=8
$$

So there are two critical points in this case, $\left(s_{1}, t_{1}\right)=(0,0)$ and $\left(s_{2}, t_{2}\right)=(8,-16)$.
c. $h(u, v)=u^{3}+v^{3}-3 u^{2}-3 v+5$. First order conditions:

$$
\begin{aligned}
& h_{u}=3 u^{2}-6 u=0 \\
& h_{v}=3 v^{2}-3=0
\end{aligned}
$$

The first equation factors as $3 u(u-2)=0$, which has two solutions $u_{1}=0$ and $u_{2}=2$. The second equation factors as well, giving $3\left(v^{2}-1\right)=0$, which has the two solutions $v_{1}=1$ and $v_{2}=-1$.
The first equation places no restrictions on the variable $v$, while the second equation places no restrictions on the variable $u$, so the critical points of the function $h(u, v)$ are

$$
\left(u_{1}, v_{1}\right)=(0,1), \quad\left(u_{1}, v_{2}\right)=(0,-1), \quad\left(u_{2}, v_{1}\right)=(2,1) \text { and }\left(u_{2}, v_{2}\right)=(2,-1) .
$$

2. Use the second derivative test to classify the critical values of the functions in the previous problem.
a. Second derivative test:

$$
\left.\begin{array}{rl}
f_{x x} & =6 \\
f_{y y} & = \\
f_{x y} & = \\
-12
\end{array}\right\} \Longrightarrow D=6 \cdot 38-144=84>0
$$

Since $D>0$ and $f_{x x}>0$, it follows that $f\left(\frac{31}{21}, \frac{4}{7}\right)=\frac{50}{21}$ is a relative minimum value. (In fact, since the second derivatives are all constant, this is the absolute minimum value.)
b. Second derivative test:

$$
\left.\begin{array}{rl}
g_{s s} & =6 s \\
g_{t t} & =6 \\
g_{s t} & =12
\end{array}\right\} \Longrightarrow D(s, t)=36 s-144
$$

Since $D(0,0)=-144<0$, the first critical point yields a saddle point on the graph of $g(s, t)$, i.e., $g(0,0)=2$ is neither max nor min. Since $D(8,-16)=144>0$ and $g_{s s}(8,-16)=48>0$, it follows that $g(8,-16)=-254$, is a relative minimum value.

Can you show that $g(8,-16)=-254$ is not the absolute minimum?
c. Second derivative test:

$$
\left.\begin{array}{rl}
h_{u u} & =6 u-6 \\
h_{v v} & =6 v \\
h_{u v} & =0
\end{array}\right\} \Longrightarrow D(u, v)=36 v(u-1) .
$$

Evaluating the discriminant at the four critical points we find that
i. $D(0,1)=-36<0$, so $h(0,1)=3$ is neither a local minimum value nor a local maximum value;
ii. $D(0,-1)=36>0$ and $h_{u u}(0,-1)=-6<0$, so $h(0,-1)=7$ is a local maximum value;
iii. $D(2,1)=36>0$ and $h_{u u}(2,1)=6>0$, so $h(2,1)=-1$ is a local minimum value; and
iv. $(D(2,-1)=-36$, so $h(2,-1)=3$ is neither a local minimum value nor a local maximum value.
3. ACME Widgets produces two competing products, type A widgets and type B widgets. The joint demand functions for these products are

$$
Q_{A}=100-3 P_{A}+2 P_{B} \text { and } Q_{B}=60+2 P_{A}-2 P_{B}
$$

and ACME's cost function is

$$
C=20 Q_{A}+30 Q_{B}+1200
$$

Find the prices that ACME should charge to maximize their profit, the corresponding output levels and the max profit. Justify your claim that the prices you found yield the absolute maximum profit.

ACME's profit function is

$$
\begin{aligned}
\Pi= & P_{A} Q_{A}+P_{B} Q_{B}-C \\
= & 100 P_{A}-3 P_{A}^{2}+2 P_{A} P_{B}+60 P_{B}+2 P_{A} P_{B}-2 P_{B}^{2} \\
& \quad-\left[20\left(100-3 P_{A}+2 P_{B}\right)+30\left(60+2 P_{A}-2 P_{B}\right)+1200\right] \\
= & -3 P_{A}^{2}+4 P_{A} P_{B}-2 P_{B}^{2}+100 P_{A}+80 P_{B}-5000 .
\end{aligned}
$$

(i) Critical point(s):

$$
\left.\begin{array}{l}
\Pi_{P_{A}}=-6 P_{A}+4 P_{B}+100=0 \\
\Pi_{P_{B}}=4 P_{A}-4 P_{B}+80=0
\end{array}\right\}
$$

Now, $\Pi_{P_{B}}=0 \Longrightarrow 4 P_{B}=4 P_{A}+80$, and plugging this into the first equation gives

$$
-6 P_{A}+\left(4 P_{A}+80\right)+100=0 \Longrightarrow-2 P_{A}+180=0 \Longrightarrow P_{A}=90 \Longrightarrow P_{B}=110
$$

So there is only one critical point, and the critical prices are $P_{A}=90$ and $P_{B}=110$, with corresponding output/demand levels $Q_{A}=50$ and $Q_{B}=20$.
(ii) Second derivative test:

$$
\left.\begin{array}{l}
\Pi_{P_{A} P_{A}}= \\
\Pi_{P_{B} P_{B}}=-6 \\
\Pi_{P_{A} P_{B}}=
\end{array}\right\} .4 \text {. }
$$

Since $D>0$ and $\Pi_{P_{A} P_{A}}=-6<0$, and the second derivatives are all constant, it follows that $\Pi(90,110)=3900$ is the absolute maximum profit.
4. An electronics retailer has determined that the number $N$ of laptops she can sell per week is

$$
N=\frac{9 x}{4+x}+\frac{20 y}{5+y}
$$

where $x$ is her weekly expenditure on radio advertising and $y$ is her weekly expenditure on internet advertising, both measured in $\$ 100$ s. Her weekly profit is $\$ 400$ per sale, less the cost of advertising.
Find the amount of money that the retailer should spend on radio and internet advertising, respectively, to maximize her weekly profit. Verify that the point you found yields a relative maximum value. What is the maximum profit?

The first step is to find the weekly profit function, $P$, which is $\$ 400$ times the number of laptops sold minus the cost of advertising:

$$
P=400 N-100(x+y)=100\left(\frac{36 x}{4+x}+\frac{80 y}{5+y}-x-y\right)
$$

Next, first-order conditions:

$$
\begin{aligned}
& P_{x}=0 \Longrightarrow 100\left(\frac{144}{(4+x)^{2}}-1\right)=0 \Longrightarrow \frac{144}{(4+x)^{2}}=1 \Longrightarrow(4+x)^{2}=144 \\
& P_{y}=0 \Longrightarrow 100\left(\frac{400}{(5+y)^{2}}-1\right)=0 \Longrightarrow \frac{400}{(5+y)^{2}}=1 \Longrightarrow(5+y)^{2}=400
\end{aligned}
$$

It follows that $4+x= \pm 12$ and $5+y= \pm 20$, and since both $x$ and $y$ must be nonnegative, we conclude that the critical numbers are $x^{*}=12-4=8$ and $y^{*}=20-5=15$.
The corresponding profit is

$$
P^{*}=100\left(\frac{288}{12}+\frac{1200}{20}-8-15\right)=6100 .
$$

To verify that this is the maximum profit, we use the second derivative test:

$$
P_{x x}=-\frac{288}{(4+x)^{3}}, P_{y y}=-\frac{800}{(5+y)^{3}} \text { and } P_{x y}=0
$$

so

$$
D(x, y)=\frac{288 \cdot 800}{(4+x)^{3}(5+y)^{3}} \Longrightarrow D(8,15)=\frac{288 \cdot 800}{12^{3} \cdot 20^{3}}>0
$$

and

$$
P_{x x}(8,15)=-\frac{288}{12^{3}}<0
$$

which shows that $P^{*}$ is a maximum.

