Study Guide 7

Solutions

1. The monthly cost function for ACME Widgets is

$$C = 0.02Q_A^2 + 0.01Q_AQ_B + 0.03Q_B^2 + 35Q_A + 28Q_B + 5000,$$

where Q_A and Q_B are the monthly outputs of type A widgets and type B widgets, respectively, measured in 100s of widgets. The cost is measured in dollars.

a. Compute the marginal cost of type A widgets and the marginal cost of type B widgets, if the monthly outputs are 25000 type A widgets and 36000 type B widgets.

Solution: $\partial C/\partial Q_A = 0.04Q_A + 0.01Q_B + 35$ and $\partial C/\partial Q_B = 0.01Q_A + 0.06Q_B + 28$. Next, remembering the units, $Q_A = 25000/100 = 250$ and $Q_B = 36000/100 = 360$, so

$$\left.\frac{\partial C}{\partial Q_A}\right|_{\substack{Q_A=250\\Q_B=360}} = 48.6 \text{ and } \left.\frac{\partial C}{\partial Q_B}\right|_{\substack{Q_A=250\\Q_B=360}} = 52.1.$$

b. Suppose that production of type A widgets is held fixed at 25000, and production of type B widgets is increased from 36000 to 36050. Use your answer to part a. to estimate the change in cost to the firm.

Solution: Approximation formula:

$$\Delta C \approx \left(\left.\frac{\partial C}{\partial Q_B}\right|_{\substack{Q_A=250\\Q_B=360}}\right) \cdot \Delta Q_B,$$

since we are assuming in this case that $\Delta Q_A = 0$. Now, $\Delta Q_B = 50/100 = 0.5$, so $\Delta C \approx (52.1)(0.5) = 26.05$.

c. Suppose that production of type A widgets is increased from 25000 to 25060, and production of type B widgets is increased from 36000 to 36040. Use your answer to part a. to estimate the change in cost to the firm.

Solution: General Approximation formula:

$$\Delta C \approx \left(\left.\frac{\partial C}{\partial Q_A}\right|_{\substack{Q_A=250\\Q_B=360}}\right) \cdot \Delta Q_A + \left(\left.\frac{\partial C}{\partial Q_B}\right|_{\substack{Q_A=250\\Q_B=360}}\right) \cdot \Delta Q_B.$$

In this case we have $\Delta Q_A = 60/100 = 0.6$ and $\Delta Q_B = 40/100 = 0.4$, so

$$\Delta C \approx \left(\left.\frac{\partial C}{\partial Q_A}\right|_{\substack{Q_A=250\\Q_B=360}}\right) \cdot \Delta Q_A + \left(\left.\frac{\partial C}{\partial Q_B}\right|_{\substack{Q_A=250\\Q_B=360}}\right) \cdot \Delta Q_B = (48.6)(0.6) + (52.1)(0.4) = 50.$$

2. The demand function for a firm's product is given by $Q = \dfrac{30\sqrt{6Y+5p_s}}{3p+5}$, where

- Q is the monthly demand for the firm's product, measured in 1000's of units,
- Y is the average monthly disposable income in the market for the firm's product, measured in 1000s of dollars,
- p_s is the average price of a substitute for the firm's product, measured in dollars,
- p is the price of the firm's product, also measured in dollars.

a. Find Q, Q_Y, Q_{p_s} and Q_p when the monthly income is \$2500 and the prices are $p_s = 17$ and $p = 15$. Round your (final) answers to two decimal places.

Solution: First, writing $Q = Q(p, p_s, Y)$, and remembering that Y is measured in \$1000s, we have

$$Q(15, 17, 2.5) = \frac{30\sqrt{15+85}}{50} = 6.$$

Next, the partial derivatives are

$$Q_Y = \frac{30}{3p+5} \cdot \frac{6}{2(6Y+5p_s)^{1/2}} = \frac{90}{(3p+5)(6Y+5p_s)^{1/2}} \implies Q_Y(15, 17, 2.5) = \frac{9}{50} = 0.18,$$

$$Q_{p_s} = \frac{30}{3p+5} \cdot \frac{5}{2(6Y+5p_s)^{1/2}} = \frac{75}{(3p+5)(6Y+5p_s)^{1/2}} \implies Q_{p_s}(15, 17, 2.5) = \frac{3}{20} = 0.15$$

and

$$Q_p = 30\sqrt{6Y+5p_s} \cdot (-1) \cdot \frac{3}{(3p+5)^2} = -\frac{90\sqrt{6Y+5p_s}}{(3p+5)^2} \implies Q_p(15, 17, 2.5) = -\frac{9}{25} = -0.36$$

b. Compute the *income-elasticity of demand* for the firm's product at the point in part **a.**

Solution:

$$\eta_{Q/Y} = Q_Y \cdot \frac{Y}{Q} \implies \eta_{Q/Y}\Big|_{\substack{p=15\\ p_s=17\\ Y=2.5}} = 0.18 \cdot \frac{2.5}{6} = 0.075$$

c. Use *linear approximation* and your answer to **a.** to estimate the change in demand for the firm's product if the price of the firm's product increases to \$16 and the price of substitutes increases to \$18, but income remains fixed.

Solution:

$$\Delta Q \approx Q_p \cdot \Delta p + Q_{p_s} \cdot \Delta p_s + Q_Y \cdot \Delta Y = -0.36 \cdot 1 + 0.15 \cdot 1 + 0.18 \cdot 0 = -0.21.$$

I.e., demand will *decrease* by about 210 units.

d. Use your answer to part **b.** to estimate the *percentage* change in demand for the firm's product if the average income increases to \$2600 while the prices stay the same as they were in part a.

Solution: The percentage change in income is $\%\Delta Y = \frac{0.1}{2.5} \cdot 100\% = 4\%$, so

$$\%\Delta Q \approx \eta_{Q/Y} \cdot \%\Delta Y = 0.075 \cdot 4\% = 0.3\%.$$

3. Find the indicated partial derivatives of the functions below.

(a) $z = 3x^2 + 4xy - 5y^2 - 4x + 7y - 2,$

$$z_{yx} = (z_y)_x = (4x - 10y + 7)_x = 4$$

$$z_{xx} = (z_x)_x = (6x + 4y - 4)_x = 6$$

(b) $F(u, v, w) = 60u^{2/3}v^{1/6}w^{1/2}$

$$\frac{\partial^2 F}{\partial w \partial u} = \frac{\partial}{\partial w}\left(\frac{\partial F}{\partial u}\right) = \frac{\partial}{\partial w}\left(40u^{-1/3}v^{1/6}w^{1/2}\right) = 20u^{-1/3}v^{1/6}w^{-1/2}$$

$$\frac{\partial^2 F}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial F}{\partial v}\right) = \frac{\partial}{\partial v}\left(10u^{2/3}v^{-5/6}w^{1/2}\right) = -\frac{50}{6}u^{2/3}v^{-11/6}w^{1/2}$$

(c) $w = x^2 z \ln(y^2 + z^3)$

$$w_{xx} = (w_x)_x = (2xz\ln(y^2 + z^3))_x = 2z\ln(y^2 + z^3)$$

$$w_{yz} = (w_y)_z = \left(\frac{2x^2yz}{y^2 + z^3}\right)_z = \frac{2x^2y(y^2 + z^3) - 3z^2(2x^2yz)}{(y^2 + z^3)^2} = \frac{2x^2y^3 - 4x^2yz^3}{(y^2 + z^3)^2}$$

$$w_{xyz} = w_{yzx} = \left(\frac{2x^2y^3 - 4x^2yz^3}{(y^2 + z^3)^2}\right)_x = \left(\frac{x^2(2y^3 - 4yz^3)}{(y^2 + z^3)^2}\right)_x = \frac{2x(2y^3 - 4yz^3)}{(y^2 + z^3)^2}$$

(d) $q(u, v) = \dfrac{u^2v - 3uv^3}{2u + 3v}$

$$\begin{aligned}
\frac{\partial^2 q}{\partial u^2} &= \frac{\partial q}{\partial u}\left(\frac{\partial q}{\partial u}\right) = \frac{\partial q}{\partial u}\left(\frac{(2uv - 3v^3)(2u + 3v) - 2(u^2v - 3uv^3)}{(2u + 3v)^2}\right) \\
&= \frac{\partial q}{\partial u}\left(\frac{2u^2v + 6uv^2 - 9v^4}{(2u + 3v)^2}\right) \\
&= \frac{(4uv + 6v^2)(2u + 3v)^{\cancel{2}1} - 4\cancel{(2u + 3v)}(2u^2v + 6uv^2 - 9v^4)}{(2u + 3v)^{\cancel{4}3}} \\
&= \frac{18v^3 + 36v^4}{(2u + 3v)^3} \qquad \left(= \frac{18v^3(2v + 1)}{(2u + 3v)^3}\right)
\end{aligned}$$