Solutions

1. The dead body of an eccentric socialite is found in a Las Vegas motel room. At 10:00 am, her body temperature was measured to be 92.4°F. Her body was left in the room for an hour and at 11:00 am her body temperature was 90.5°F. The room itself was kept at a constant temperature of 72°F.

Use *Newton's law of cooling* to estimate the time of the socialite's death.

(*) See Example 3 in section 15.6.

Newton's law of cooling (or heating), says that the the rate of change in the temperature of a body immersed in a medium of constant ambient temperature is proportional to the difference between the temperature of the body T and the ambient temperature A. If $T(t)$ is the temperature of the body at time t, then Newton's law can be expressed as

$$\frac{dT}{dt} = k(T - A),$$

where k is the unknown constant of proportionality.† To solve this differential equation, we separate, integrate and solve for T:

$$\begin{aligned}
\text{Separate}: & \ \frac{dT}{T-A} = k\,dt \\
\text{Integrate}: & \ \int \frac{dT}{T-A} = \int k\,dt \implies \ln|T-A| = kt + C \\
\text{Solve for T}: & \ |T-A| = e^{kt+C} = e^C \cdot e^{kt} \implies T - A = \pm e^C \cdot e^{kt} \implies T = A + \alpha e^{kt},
\end{aligned}$$

where $\alpha = \pm e^C$.

Next, we use the given data. First, the constant ambient temperature is $A = 72$°F, so $T = 72 + \alpha e^{kt}$. Second, we'll let $t = 0$ correspond to 10:00 am, so

$$92.4 = T(0) = 72 + \alpha e^{k\cdot 0} = 72 + \alpha \implies \alpha = 92.4 - 72 = 20.4.$$

Next, measuring time in hours, 11:00 am corresponds to $t = 1$, so

$$90.5 = T(1) = 72 + 20.4e^{k\cdot 1} = 72 + 20.4e^k \implies e^k = \frac{90.5 - 72}{20.4} \implies k = \ln\frac{18.5}{20.4}$$

Finally, assuming that the socialite's temperature was normal (98.6°F) at her time of death (t_d) , we find that

$$98.6 = T(t_d) = 72 + 20.4e^{kt_d} \implies e^{kt_d} = \frac{26.6}{20.4} \implies kt_d = \ln\frac{26.6}{20.4} \implies t_d = \frac{\ln\frac{26.6}{20.4}}{\ln\frac{18.5}{20.4}} \approx -2.714.$$

This means that the socialite died 2.714 hours *before* 10:00 am—at roughly 7:17 am.

2. The population of a tropical island grows at a rate that is proportional to the *third root* ($\sqrt[3]{\ }$) of its size. In 1950, the island's population was 1728 and in 1980, the island's population was 2744. What will the island's population be in 2020?

†The constant k depends on the physical properties of the body, in particular its heat conductance.

First, translate the description of the growth rate into a differential equation. If $P(t)$ is the size of the population at time t (in years), then the description above leads to the differential equation

$$\frac{dP}{dt} = k\sqrt[3]{P},$$

where k is the (unknown) constant of proportionality.

Separate the variables: $\dfrac{dP}{P^{1/3}} = k\,dt.$

Integrate both sides: $\displaystyle\int P^{-1/3}\,dP = \int k\,dt \quad\Longrightarrow\quad \frac{3}{2}P^{2/3} = kt + C.$

Solve for P: Multiplying by $2/3$ gives $P^{2/3} = kt + C$, because the factor of $2/3$ is absorbed by both k and C. Next, raise both sides to the power $3/2$ to see that

$$P = (kt + C)^{3/2}.$$

Solve for the parameters k and C: First set $t = 0$ for the year 1950, so

$$1728 = P(0) = (0 + C)^{3/2} = C^{3/2} \quad\Longrightarrow\quad \boxed{C = 1728^{2/3} = 144}.$$

Next, the year 1980 corresponds to $t = 30$, so

$$2744 = P(3) = (30k + 144)^{3/2} \quad\Longrightarrow\quad 30k + 144 = 2744^{2/3} = 196 \quad\Longrightarrow\quad \boxed{k = \frac{26}{15}}.$$

Thus $P(t) = \left(\frac{26}{15}t + 144\right)^{3/2}$ and in 2020 the population will be

$$P(70) = \left(\frac{26}{15}\cdot 70 + 144\right)^{3/2} \approx 4322.$$

3. The income-elasticity of monthly demand (q) for a price-controlled good is assumed to be proportional to the natural logarithm of average monthly disposable income (y) in the market for that good. When $y = 2500$, the demand is $q = 500$ and when $y = 2000$, the demand is $q = 350$. What is the predicted monthly demand for this good if monthly disposable income decreases to $y = 1500$?

The income-elasticity of demand is $\eta = \dfrac{dq}{dy}\cdot\dfrac{y}{q}$, and if this is proportional to the natural logarithm of income, we obtain the separable differential equation

$$\frac{dq}{dy}\cdot\frac{y}{q} = k\ln y \;\Longrightarrow\; \frac{dq}{q} = k\frac{\ln y}{y}\,dy,$$

with k being the unknown constant of proportionality (as usual). Integrating both sides gives

$$\int\frac{dq}{q} = k\int\frac{\ln y}{y}\,dy \;\Longrightarrow\; \ln q = k\frac{(\ln y)^2}{2} + C = k(\ln y)^2 + C,$$

using the substitution $u = \ln y,\ du = \frac{1}{y}\,dy$ for the right-hand integral and absorbing the factor of $1/2$ into k.[‡] Exponentiating both sides gives

$$q = e^{k(\ln y)^2 + C} = e^C \cdot e^{k(\ln y)^2} = Ae^{k(\ln y)^2},$$

and it remains to use the given data to solve for A and k. The data $q(2500) = 500$ and $q(2000) = 350$ gives a pair of equations for A and k:

$$\left.\begin{array}{rcl} Ae^{k(\ln 2000)^2} & = & 350 \\ Ae^{k(\ln 2500)^2} & = & 500 \end{array}\right\} \implies \frac{\cancel{A}e^{k(\ln 2000)^2}}{\cancel{A}e^{k(\ln 2500)^2}} = \frac{350}{500} \implies e^{k\left[(\ln 2000)^2 - (\ln 2500)^2\right]} = 0.7$$

Next, take the natural logarithm of both sides to find k:

$$k\left[(\ln 2000)^2 - (\ln 2500)^2\right] = \ln 0.7 \implies k = \frac{\ln 0.7}{(\ln 2000)^2 - (\ln 2500)^2} \approx 0.103625.$$

To find A, use one of the two equations:

$$Ae^{k(\ln 2000)^2} = 350 \implies A = 350e^{-k(\ln 2000)^2} \approx 0.879.$$

Finally, we can predict the demand when income decreases to $y = 1500$:

$$q(1500) = Ae^{k(\ln 1500)^2} \approx 0.879e^{0.103625(\ln 1500)^2} \approx 224.35.$$

[‡]Why did I not write $\ln|q|$ and $\ln|y|$ here?