Solutions

1. What are the *Producers' surplus* and *Consumers' surplus* for the market with supply function

$$p = 0.05q^2 + 3q + 5$$

and demand function

$$p = 100 - 0.75q.$$

First we find the market equilibrium price and demand:

$$\mathit{supply} = \mathit{demand} \implies 0.05q^2 + 3q + 5 = 100 - 0.75q \implies 0.05q^2 + 3.75q - 95 = 0$$

From the quadratic formula, we find that

$$q = \frac{-3.75 \pm \sqrt{3.75^2 - 4 \cdot 0.2 \cdot (-95)}}{0.1} = \begin{cases} \dfrac{-3.75 + 5.75}{0.1} & = & 20 \\ & \mathit{or} & \\ \dfrac{-3.75 - 5.75}{0.1} & = & -95 \end{cases}$$

So, the equilibrium demand is $q^* = 20$ *(because demand must be positive) and the equilibrium price is* $p^* = 100 - 0.75q^* = 85$.

Next we compute the consumers' and producers' surplus:

$$CS = \int_0^{q^*} (\mathit{demand} - p^*)\, dq = \int_0^{20} 100 - 0.75q - 85\, dq = 15q - \frac{3}{8}q^2 \Big|_0^{20} = 150$$

and

$$PS = \int_0^{q^*} (p^* - \mathit{supply})\, dq = \int_0^{20} 85 - (0.05q^2 + 3q + 5)\, dq = 80q - \frac{3}{2}q^2 - \frac{1}{60}q^3 \Big|_0^{20} \approx 866.67$$

2. Find the average value of the function $f(x) = \dfrac{x^4 - 1}{x^2}$ on the interval $[1, 3]$.

The average value of a function on an interval is equal to the definite integral of the function on the interval, divided by the length of the interval, i.e.,

$$\mathit{avg}(f) = \frac{1}{b - a} \int_a^b f(x)\, dx.$$

In this case, we get

$$\mathit{avg}(f) = \frac{1}{3 - 1} \int_1^3 \frac{x^4 - 1}{x^2}\, dx = \frac{1}{2} \int_1^3 x^2 - x^{-2}\, dx = \frac{1}{6}x^3 + \frac{1}{2}x^{-1} \Big|_1^3 = \frac{28}{6} - \frac{4}{6} = 4$$

3. Use the table of integral formulas in Appendix B in the textbook to help compute the integrals below.

a. $\displaystyle\int \frac{4\,dx}{5x\sqrt{x^2+9}} = \frac{4}{5}\cdot\frac{1}{3}\ln\left|\frac{\sqrt{x^2+9}-3}{x}\right| + C = \frac{4}{15}\ln\left|\frac{\sqrt{x^2+9}-3}{x}\right| + C$

Formula #28, with $a = 3$.

b. $\displaystyle\int \frac{2e^{2x}\,dx}{\sqrt{9+4e^x}} = 2\int \frac{e^x e^x\,dx}{\sqrt{9+4e^x}} = 2\int \frac{u\,du}{\sqrt{9+4u}}$

$$= 2\cdot\frac{2(4u-18)\sqrt{9+4u}}{48} + C = \frac{(2e^x-9)\sqrt{9+4e^x}}{6} + C$$

First, note that $e^{2x} = e^x e^x$ and substitute $u = e^x,\ du = e^x\,dx$, then use formula #15 with $a = 9$ and $b = 4$.

c. $\displaystyle\int_0^{10} 200t^2e^{-0.06t}\,dt = 200\left(\left.\frac{t^2e^{-0.06t}}{-0.06}\right|_0^{10} - \frac{2}{-0.06}\int_0^{10} te^{-0.06t}\,dt\right)$

$$= 200\left(\left(-\frac{100e^{-0.6}}{0.06} - 0\right) + \frac{2}{0.06}\cdot\left.\frac{e^{-0.06t}}{0.06^2}(-0.06t-1)\right|_0^{10}\right)$$

$$= 200\left(-\frac{100e^{-0.6}}{0.06} - \frac{3.2}{0.06^3}e^{-0.6} + \frac{2}{0.06^3}\right) \approx 42806.0884$$

Formula #39 with $n = 2$ and $a = -0.06$, followed by formula #38 (with $a = -0.06$ again).

d. $\displaystyle\int_0^3 \frac{2\,dv}{\sqrt{v^2+16}} = 2\ln\left|v+\sqrt{v^2+16}\right|\Big|_0^3 = 2(\ln(3+5) - \ln(0+4)) = 2\ln 2$

Formula #27, with $a = 4$.

e. $\displaystyle\int 5x^3\ln x\,dx = \frac{5x^4\ln x}{4} - \frac{5x^4}{16} + C$

Formula #42, with $n = 3$.

f. $\displaystyle\int_0^2 \frac{3+5x}{2+7x}\,dx = 3\int_0^2 \frac{1}{2+7x}\,dx + 5\int_0^2 \frac{x}{2+7x}\,dx = \left(\left.\frac{3}{7}\ln|2+7x|\right|_0^2\right) + \left(\left.\frac{5x}{7} - \frac{10}{49}\ln|2+7x|\right|_0^2\right)$

$$= \frac{3}{7}(\ln 16 - \ln 2) + \left(\frac{10}{7} - \frac{10}{49}\ln 16\right) - \left(0 - \frac{10}{49}\ln 2\right) = \frac{10}{7} + \frac{11}{49}\ln 8$$

Formulas #2 and #3, with $a = 2$ and $b = 7$.

4. Compute the present value of the continuous annuity that pays at the continuous rate $f(t) = 250t$ for $T = 20$ years, where the constant interest rate is $r = 4.75\%$.

See section 15.3 in the text, and formula #38.

$$\begin{aligned}\text{Present Value} &= \int_0^T f(t)e^{-rt}\,dt \\ &= \int_0^{20} 250te^{-0.0475t}\,dt \\ &= \frac{250e^{-0.0475t}}{(0.0475)^2}(-0.0475t-1)\Big|_0^{20} \approx 27241.55.\end{aligned}$$

5. Let $y = f(x)$ satisfy (i) $\dfrac{dy}{dx} = 3xy^2$ and (ii) $y(1) = 2$. Find the function $f(x)$.

Separate: $\dfrac{dy}{y^2} = 3x\,dx$.

Integrate: $\displaystyle\int \frac{dy}{y^2} = \int 3x\,dx \quad \Longrightarrow \quad -\frac{1}{y} = \frac{3x^2}{2} + C$. (This is the *implicit* solution.)

Solve for y: $y = \dfrac{2}{C - 3x^2}$.

Solve for C: $y(1) = 2 \implies 2 = \dfrac{2}{C-3} \implies C - 3 = 1 \implies C = 4$.

Solution: $y = \dfrac{2}{4 - 3x^2}$.