Solutions

1. Compute the indefinite integrals below.

(a) $\displaystyle\int \frac{3x\,dx}{\sqrt[3]{x^2+1}} = \dots$

(*) Substitute $u = x^2+1 \implies du = 2x\,dx \implies 3x\,dx = \frac{3}{2}\,du$

$$\int \frac{3x\,dx}{\sqrt[3]{x^2+1}} = \int \frac{3/2\,du}{\sqrt[3]{u}} = \frac{3}{2}\int u^{-1/3}\,du = \frac{3}{2}\cdot\frac{u^{2/3}}{2/3} + C$$
$$= \frac{9}{4}u^{2/3} + C = \frac{9}{4}\left(x^2+1\right)^{2/3} + C$$

(b) $\displaystyle\int (x^2+2x)(x^3+3x^2-1)^3\,dx = \dots$

(*) Substitute $u = x^3+3x^2-1 \implies du = (3x^2+6x)\,dx \implies (x^2+2x)\,dx = \frac{1}{3}\,du$

$$\int (x^2+2x)(x^3+3x^2-1)^3\,dx = \frac{1}{3}\int u^3\,du = \frac{1}{12}u^4 + C = \frac{1}{12}\left(x^3+3x^2-1\right)^4 + C$$

(c) $\displaystyle\int 1000e^{-0.05t}\,dt = \dots$

(*) Substitute $u = -0.05t \implies du = -0.05\,dy \implies dt = -20\,du$

$$\int 1000e^{-0.05t}\,dt = -20,000\int e^u\,du = -20,000e^u + C = -20,000e^{-0.05t} + C$$

(d) $\displaystyle\int \frac{t^2+5}{3t+1}\,dt = \dots$

(*) This one can be done using *long division* (of polynomials)

$$\frac{t^2+5}{3t+1} = \frac{1}{3}t - \frac{1}{9} + \frac{46/9}{3t+1}$$

to simplify the integrand before integration. The integration will require substituting $u = 3t+1$. Alternatively, you can make this substitution in the original integral. This also requires solving for t (in the numerator):

$$u = 3t+1 \implies t = \frac{1}{3}(u-1) \text{ and } u = 3t+1 \implies du = 3\,dt \implies dt = \frac{1}{3}\,du.$$

Using the second approach, we see that

$$\int \frac{t^2+5}{3t+1}\,dt = \frac{1}{3}\int \frac{\left(\frac{1}{3}(u-1)\right)^2+5}{u}\,du = \frac{1}{3}\int \frac{\frac{1}{9}u^2 - \frac{2}{9}u + \frac{46}{9}}{u}\,du$$
$$= \frac{1}{27}\int u - 2 + \frac{46}{u}\,du = \frac{1}{54}u^2 - \frac{2}{27}u + \frac{46}{27}\ln|u| + C$$
$$= \frac{1}{54}(3t+1)^2 - \frac{2}{27}(3t+1) + \frac{46}{27}\ln|3t+1| + C$$

(*) This can be further simplified, if desired:

$$= \frac{1}{54}(9t^2 + 6t + 1) - \frac{2}{9}t - \frac{2}{27} + \frac{46}{27}\ln|3t+1| + C$$
$$= \frac{1}{6}t^2 - \frac{1}{9}t + \frac{46}{27}\ln|3t+1| + C,$$

which is the answer your would get if you took the long-division approach. Note that the constants $2/27$ and $1/54$ were 'absorbed' by the constant of integrations C.

2. Find the function $y = g(x)$, given that $y'' = x^2 - 1$, $g'(1) = 2$ and $g(1) = 2$.

First solve one initial value problem to find y', by integrating y'':

$$y' = \int y''\,dx = \int x^2 - 1\,dx = \frac{x^3}{3} - x + C_1.$$

Next, solve for C_1 using the initial value for y':

$$2 = y'(1) = \frac{1^3}{3} - 1 + C_1 \implies C_1 = 2 + 1 - \frac{1}{3} = \frac{8}{3}.$$

So, $y' = \frac{x^3}{3} - x + \frac{8}{3}$, and now we repeat the process to find $y = g(x)$.

$$g(x) = \int y'\,dx = \int \frac{x^3}{3} - x + \frac{8}{3}\,dx = \frac{x^4}{12} - \frac{x^2}{2} + \frac{8x}{3} + C_2.$$

Use the initial data for $g(x)$ to solve for C_2:

$$2 = g(1) = \frac{1}{12} - \frac{1}{2} + \frac{8}{3} + C_2 \implies C_2 = -\frac{1}{4},$$

giving the final solution $g(x) = \frac{x^4}{12} - \frac{x^2}{2} + \frac{8x}{3} - \frac{1}{4}$.

(*) The next two problems are solved using the same idea. If $y = f(x) + C$, then even if we don't know C, we can compute Δy — the change in the value of y — using the formula

$$\Delta y = y(x_2) - y(x_1) = f(x_2) + C - (f(x_1) + C) = f(x_2) - f(x_1).$$

I.e., the change in y does not depend on the constant. This is useful when we know the derivative y, but not y itself. This is essentially the same as computing a *definite* integral, which we will learn next.

3. A firm's marginal revenue and marginal cost functions are

$$\frac{dr}{dq} = 100 - \sqrt{3q + 10} \quad \text{and} \quad \frac{dc}{dq} = 0.2q + 65,$$

respectively. How will the firm's **profit** change if output is increased from $q = 30$ to $q = 53$?

The profit function is given by $\pi = r - c$, so the derivative of the profit function (in this problem) is given by

$$\frac{d\pi}{dq} = \frac{dr}{dq} - \frac{dc}{dq} = 100 - \sqrt{3q+10} - (0.2q + 65) = 35 - 0.2q - \sqrt{3q+10}.$$

This means that the profit function is given by

$$\begin{aligned}
\pi &= \int 35 - 0.2q - \sqrt{3q+10}\, dq = \int 35 - 0.2q\, dq - \int \sqrt{3q+10}\, dq \\
&= 35q - 0.1q^2 - \frac{1}{3}\int u^{1/2}\, du = 35q - 0.1q^2 - \frac{1}{3} \cdot \frac{u^{3/2}}{3/2} + C \\
&= 35q - 0.1q^2 - \frac{2}{9}(3q+10)^{3/2} + C,
\end{aligned}$$

using the substitution $u = 3q + 10 \implies du = \frac{1}{3}\, dq$ in the last integral on first line.

It follows that the change in the firm's profit is

$$\begin{aligned}
\Delta\pi &= \pi(53) - \pi(30) \\
&= \left(35 \cdot 53 - 0.1 \cdot 2809 - \frac{2}{9} \cdot 169^{3/2}\right) - \left(35 \cdot 30 - 0.1 \cdot 900 - \frac{2}{9} \cdot 100^{3/2}\right) \\
&= 1085.8\overline{777} - 737.7\overline{777} = 348.10
\end{aligned}$$

4. The marginal propensity to consume of a small nation is given by

$$\frac{dC}{dY} = \frac{9Y + 10}{10Y + 1},$$

where consumption C and national income Y are both measured in billions of dollars. Find the total change in national consumption and saving, if income increases from \$10 billion to \$15 billion.

Using the substitution

$$u = 10Y + 1 \implies Y = \frac{1}{10}(u - 1) \text{ and } dY = \frac{1}{10}\, du$$

to compute the integral, we see that the nation's consumption function is

$$\begin{aligned}
C &= \int \frac{9Y+10}{10Y+1}\, dY = \frac{1}{10}\int \frac{\frac{9}{10}(u-1) + 10}{u}\, du \\
&= \frac{1}{100}\int \frac{9u + 91}{u}\, du = \frac{1}{100}\int 9 + \frac{91}{u}\, du \\
&= \frac{9}{100}u + \frac{91}{100}\ln|u| + K \\
&= \frac{9}{100}(10Y + 1) + \frac{91}{100}\ln|10Y + 1| + K \\
&= 0.9Y + 0.91\ln|10Y + 1| + K.
\end{aligned}$$

Observe that the constant 9/100 in the fourth line was 'absorbed' by the constant of integration K. It follows that the change in consumption is

$$\Delta C = C(15) - C(10) = 0.9 \cdot 15 + 0.91 \ln|151| - 0.9 \cdot 10 - 0.91 \ln|101| \approx 4.866,$$

and therefore the change in saving is

$$\Delta S = \Delta Y - \Delta C \approx 5 - 4.866 = 0.134.$$

In words, if income increases from $10 billion to $15 billion, consumption will increase by about $4.866 billion and saving will increase by about $134 million.

(*) I also used the fact from economics that $Y = C + S$, so $S = Y - C$ and therefore $\Delta S = \Delta Y - \Delta C$.