(*) The Envelope Theorem and constrained optimization

As we saw in class on Monday (and Friday, and in Supplemental Notes #5), one of the important *applications* of the envelope theorem is to constrained optimization.

Suppose that we solve the problem of finding the maximum/minimum value of the (objective) function $f(x, y, z)$ subject to a constraint of the form $g(x, y, z) = c$ using the method of Lagrange multipliers. Briefly, we form the Lagrangian

$$F(x, y, z, \lambda) = f(x, y, z) - \lambda(g(x, y, z) - c)$$

and find the critical point $(x^*, y*, z^*, \lambda^*)$ by solving the system of equations

$$F_x = 0, \quad F_y = 0, \quad F_z = 0 \quad \text{and} \quad F_\lambda = 0.$$

Since $F_\lambda = -(g(x, y, z) - c)$, it follows that $g(x^*, y^*, z^*) - c = 0$, and therefore

$$F^* = F(x^*, y^*, z^*, \Lambda^*) = f(x^*, y^*, z^*) - \lambda^* \overbrace{(g(x^*, y^*, z^*) - c)}^{=0} = f(x^*, y^*, z^*) = f^*.$$

That is to say the constrained optimum f^* is always the same as the critical value of the Lagrangian F^*.

This is useful, because it allows us to easily find the rate of change of f^* with respect to the constraint c. Specifically, we can use the envelope theorem as follows.

$$\overbrace{\frac{df^*}{dc} = \frac{dF^*}{dc}}^{f^*=F^*} \overbrace{= \left.\frac{\partial F}{\partial c}\right|_{\substack{x=x^*\\y=y^*\\z=z^*\\\lambda=\lambda^*}}}^{\text{Envelope Theorem}} = \lambda\Big|_{\substack{x=x^*\\y=y^*\\z=z^*\\\lambda=\lambda^*}} = \lambda^*$$

because

$$\frac{\partial F}{\partial c} = \frac{\partial}{\partial c}\big(f(x, y, z) - \lambda(g(x, y, z) - c)\big) = \frac{\partial}{\partial c}\big(f(x, y, z) - \lambda g(x, y, z) + \lambda c\big) = \lambda.$$

In words, *the critical value of the multiplier λ^* gives the rate of change of the constrained optimum f^* with respect to the constraining parameter c.* In practical terms, linear approximation tells us that if the constraining parameter changes by Δc, then the constrained optimum changes by $\Delta f^* \approx (df^*/dc) \cdot \Delta c = \lambda^* \cdot \Delta c$. In particular, if $\Delta c = 1$, then $\Delta f^* \approx \lambda^*$.

This is perhaps easier to understand in specific examples.

(a) In the utility maximization problem — maximize the utility function $U(x, y, z)$ subject to the budget constraint $p_x x + p_y y + p_z z = B$ — λ^* is (approximate) amount by which max utility U^* increases if the budget increases by $\Delta B = \$1.00$. In this example, λ^* is *marginal utility of* \$1.00.

(b) In the cost minimization problem — minimize the cost $C(k, l)$ of producing an output of $Q(k, l) = q$ — λ^* is the (approximate) amount by which cost increases when output increases by one unit. I.e., λ^* is the *marginal cost* in this case.

(*) **But...**

The interpretation of λ^* given above is just one (important) application of the envelope theorem. The envelope theorem is much more general than that, as the example below illustrates.

Consider the utility maximization problem that we did on Friday:

> *Find the quantities x, y and z of goods 1, 2 and 3 that maximize a consumer's monthly utility*
> $$U(x, y, z) = 6 \ln x + 9 \ln y + 10 \ln z$$
> *if the average prices of goods 1, 2 and 3 are $p_x = 10$, $p_y = 20$ and $p_z = 25$, respectively and the consumer's monthly budget is $B = \$5000$.*

The Lagrangian in this case is

$$F(x, y, z, \lambda) = U(x, y, z) - \lambda(p_x x + p_y y + p_z z - B) = 6 \ln x + 9 \ln y + 10 \ln z - \lambda(10x + 20y + 25z - 5000),$$

and the critical point we found was $(x^*, y^*, z^*, \lambda^*) = (120, 90, 80, 1/200)$. If the price p_x of good #1 increases from $p_x = 10.00$ to $p_x = 10.50$ (and nothing else changes), then we can expect the consumer's max utility to decrease. To estimate the amount by which U^* changes, we can use linear approximation:

$$\Delta U^* \approx \frac{dU^*}{dp_x} \cdot \Delta p_x.$$

We know that $\Delta p_x = 0.50$, and we can use the envelope theorem to find dU^*/dp_x in the same way that we found dU^*/dB. Specifically,

$$\overbrace{\frac{dU^*}{dp_x} = \frac{dF^*}{dp_x}}^{U^* = F^*} \overbrace{= \left.\frac{\partial F}{\partial p_x}\right|_{\substack{x=x^* \\ y=y^* \\ z=z^* \\ \lambda=\lambda^*}}}^{\text{envelope theorem}} = -\lambda^* x^* = -\frac{3}{5},$$

because

$$\frac{\partial F}{\partial p_x} = \frac{\partial}{\partial p_x}\big(U(x, y, z) - \lambda(p_x x + p_y y + p_z z - B)\big) = -\lambda x.$$

I.e., if p_x increases to 10.50, the consumers (max) utility will change by (approximately) $-(3/5)(0.5) = -0.3$ utils.