(*) **Utility maximization**

A consumer's utility function $U(x, y, z)$ measures the *usefulness* to said consumer of consuming quantities x, y and z of goods 1, 2 and 3. We first observe that consumers cannot consume negative quantities, so U is only defined for $x > 0, y > 0$ and $z > 0$.

Two further (fairly standard) assumptions about such functions are (i) utility increases the more one consumes, but (ii) that the more one has consumed of a certain good, the less additional utility is derived from consuming one more unit of that good. These assumptions translate to conditions on the partial derivatives U_x, U_y and U_z,[†] namely that they are all *positive* on the one hand and all *decreasing* on the other.

One simple realization of these assumption is a utility function of the form

$$U(x, y, z) = a \ln x + b \ln y + c \ln z,$$

where the parameters a, b and c are all positive.

Since consumers are usually constrained by limited resources, a typical optimization problem considered by economists is the problem of maximizing utility subject to a budget (or income) constraint. I.e., suppose that the average prices of the goods 1, 2 and 3 are p_x, p_y and p_z, respectively and that the consumer has a budget of B dollars that they can spend on these goods. The (rational) consumer's goal is then to maximize the utility function

$$U(x, y, z) = a \ln x + b \ln y + c \ln z,$$

subject to the constraint

$$\overbrace{p_x \cdot x}^{\text{\$ spent on 1}} + \overbrace{p_y \cdot y}^{\text{\$ spent on 2}} + \overbrace{p_z \cdot z}^{\text{\$ spent on 3}} = B.$$

Example. Find the quantities x, y and z of goods 1, 2 and 3 that maximize a consumer's monthly utility

$$U(x, y, z) = 6 \ln x + 9 \ln y + 10 \ln z$$

if the average prices of goods 1, 2 and 3 are $p_x = 10$, $p_y = 20$ and $p_z = 25$, respectively and the consumer's monthly budget is $B = \$5000$, and find the consumer's maximum utility.

This is a constrained optimization problem, and we will use the method of Lagrange multipliers to solve it.[‡]

(i) *Write down the (explicit) constraint and form the Lagrangian function.*

The constraint in this case is

$$10x + 20y + 25z = 5000$$

so the Lagrangian is

$$F(x, y, z, \lambda) = U(x, y, z) - \lambda(p_x x + p_y y + p_z z - B) = 6 \ln x + 9 \ln y + 10 \ln z - \lambda(10x + 20y + 25z - 5000).$$

[†]These are the *marginal utilities* of goods 1, 2 and 3, respectively.

[‡]Because (a) the algebra is actually easier and (b) the critical value of the multiplier has an important interpretation in this problem.

(ii) *Find the critical point(s):*

$$\begin{aligned}
F_x = 0 &\implies & \frac{6}{x} - 10\lambda = 0 &\implies & \lambda = \frac{6}{10x} = \frac{3}{5x} \\
F_y = 0 &\implies & \frac{9}{y} - 20\lambda = 0 &\implies & \lambda = \frac{9}{20y} \\
F_z = 0 &\implies & \frac{10}{z} - 25\lambda = 0 &\implies & \lambda = \frac{10}{25z} = \frac{2}{5z} \\
F_\lambda = 0 &\implies & 10x + 20y + 25z = 5000. & &
\end{aligned}$$

Recall that in these problems the equation $F_\lambda = 0$ is always equivalent to the original constraint.[§] From the first and second equations we find that

$$\lambda = \frac{3}{5x} = \frac{9}{20y} \implies 60y = 45x \implies y = \frac{45}{60}x = \frac{3}{4}x,$$

and comparing the first and third equations gives

$$\lambda = \frac{3}{5x} = \frac{2}{5z} \implies 15z = 10x \implies z = \frac{10}{15}x = \frac{2}{3}x.$$

Substituting these expressions for y and z into the constraint (the equation $F_\lambda = 0$), we can find the critical x-value, x^*:

$$10x + 20\left(\frac{3}{4}x\right) + 25\left(\frac{2}{3}x\right) = 5000 \implies 125x = 15000 \implies x^* = 120.$$

Next we find that the critical y- and z-values are $y^* = \frac{3}{4}x^* = 90$ and $z^* = \frac{2}{3}x^* = 80$. Finally we find that the critical value of the multiplier is $\lambda^* = \frac{3}{5x^*} = \frac{1}{200}$ and that the consumer's maximum utility subject to the budget constraint is

$$U^* = U(x^*, y^*, z^*) = 6\ln 120 + 9\ln 90 + 10\ln 80 \approx 113.04.$$

Clearly, as the budget increases (or decreases) the quantities of the goods the consumer can buy also increase (or decrease), and therefore the maximum utility will increase (or decrease) with the budget. I.e., the maximum utility, U^*, is a function of the budget B, and it would be useful to know the rate, dU^*/dB, at which U^* changes with respect to B. To compute this, we use the *envelope theorem* together with the following important observation: the critical value of the Lagrangian is equal to the constrained critical value of the utility function, i.e., $F^* = U^*$, because

$$F^* = F(x^*, y^*, z^*, \lambda^*) = U(x^*, y^*, z^*) - \lambda^* \overbrace{(10x^* + 20y^* + 25z^* - 5000)}^{=0 \text{ because } F_\lambda = 0 \text{ at critical point}} = U^*.$$

This means that $dU^*/dB = dF^*/dB$ and

$$\left.\frac{dF^*}{dB}\right|_{B=5000} = \left.\frac{\partial F}{\partial B}\right|_{\substack{x=x^*\\y=y^*\\z=z^*\\\lambda=\lambda^*}} = \left.\frac{\partial}{\partial B}\Big(U(x,y,z) - \lambda(10x + 20y + 25z - B)\Big)\right|_{\substack{x=x^*\\y=y^*\\z=z^*\\\lambda=\lambda^*}} = \left.\lambda\right|_{\substack{x=x^*\\y=y^*\\z=z^*\\\lambda=\lambda^*}} = \lambda^*.$$

[§]See section 3 of SN 5.

I.e., the rate of change of maximum utility with respect the consumer's budget is equal to the critical value of the multiplier:

$$\frac{dU^*}{dB} = \lambda^* \qquad \left(= \frac{1}{200} \text{ in this case}\right)$$

The critical value of the multiplier, λ^*, is sometimes called the *marginal utility of a dollar* (or of money, more generally), since it the approximate amount by which (maximum) utility will increase if the budget increases by \$1.

(*) **Quiz 8**

Find the minimum value of the function $f(x, y) = 2x^2 + 3y^2$ subject to the constraint $x + y = 10$.

Solution:

(i) Lagrangian: $F(x, y, \lambda) = 2x^2 + 3y^2 - \lambda(x + y - 10)$

(ii) Critical point:

$$\begin{array}{lll} (a)\ F_x = 0 \implies & 4x - \lambda = 0 \implies & \lambda = 4x \\ (b)\ F_y = 0 \implies & 6y - \lambda = 0 \implies & \lambda = 6y \\ (c)\ F_\lambda = 0 \implies & x + y = 10 & \end{array}$$

Comparing (a) to (b) gives $6y = 4x$, so $y = 4x/6 = 2x/3$. Substituting this into (c) gives

$$x + y = 10 \implies x + \frac{2}{3}x = 10 \implies 3x + 2x = 30 \implies x^* = 6,$$

and the condition $y = 2x/3$ means that $y^* = 4$.

The desired minimum value is therefore $f^* = f(x^*, y^*) = 2 \cdot 6^2 + 3 \cdot 4^2 = 120$.