(*) **Linear approximation with partial derivatives**

The *partial derivatives* of a function $w = f(x, y)$ are defined by the same limits,

$$\frac{\partial w}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x},$$
$$\frac{\partial w}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y},$$

that define an ordinary derivative of a function of one variable. This means that the argument justifying the idea of linear approximation for a function of one variable can be extended to functions of several variables.

Specifically, given the function $w = f(x, y)$ and a point (x_0, y_0), it follows from the definition that

$$\left.\frac{\partial w}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

and therefore if $\Delta x \approx 0$, then

$$\left.\frac{\partial w}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}} \approx \frac{\overbrace{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}^{\Delta w}}{\Delta x}$$

and multiplying both sides of this approximate equality by Δx shows that if $\Delta x \approx 0$, then

$$\Delta w \approx \left.\frac{\partial w}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}} \cdot \Delta x.$$

Implicit in this version of linear approximation is the assumption that only the variable x is changing, i.e., y is held fixed at y_0.

And of course we can play the same game with the variable y, holding x fixed at x_0. I.e., if

$$\Delta w = f(x_0, y_0 + \Delta y) - f(x_0, y_0)$$

and $\Delta y \approx 0$, then

$$\Delta w \approx \left.\frac{\partial w}{\partial y}\right|_{\substack{x=x_0\\ y=y_0}} \cdot \Delta y.$$

(*) **Example.**

Suppose that the demand function for a monopolistic firm's good is given by

$$q = 3 \ln \left(4Y^{1/2} - 0.5p\right),$$

where

- q is the monthly demand for the firm's good, measured in 1000s of units,
- p is the price/unit of the firm's good, and
- Y is the average monthly household income in the market for the firm's good.

When the price is $p_0 = 10$ and the average monthly income is \$4000.00 (so $Y_0 = 4$), the demand for the firm's good is

$$q_0 = 3\ln(4 \cdot 4^{1/2} - 0.5 \cdot 10) \approx 3.296,$$

which translates to about $3,296$ units/month.

The partial derivative of q with respect to p and Y at the point above are

$$\left.\frac{\partial q}{\partial p}\right|_{\substack{p=10\\Y=4}} = \left.\frac{3\cdot(-0.5)}{4Y^{1/2} - 0.5p}\right|_{\substack{p=10\\Y=4}} = \frac{3\cdot(-0.5)}{4\cdot 4^{1/2} - 0.5\cdot 10} = \frac{-1.5}{3} = -0.5$$

and

$$\left.\frac{\partial q}{\partial Y}\right|_{\substack{p=10\\Y=4}} = \left.\frac{3\cdot(4\cdot\frac{1}{2}Y^{-1/2})}{4Y^{1/2} - 0.5p}\right|_{\substack{p=10\\Y=4}} = \frac{3\cdot(4\cdot\frac{1}{2}4^{-1/2})}{4\cdot 4^{1/2} - 0.5\cdot 10} = \frac{3}{3} = 1.$$

If the firm raises its price to $p_1 = 10.4$ and the income remains the same, then $\Delta p = 0.4$ and, using linear approximation, we see that the change in demand will be

$$\Delta q \approx \left.\frac{\partial q}{\partial p}\right|_{\substack{p=10\\Y=4}} \cdot \Delta p = -0.5 \cdot 0.4 = -0.2,$$

i.e., demand for the firm's good will *decrease* by about 200 units/month.

On the other hand, if the firm keeps their price fixed at $p_0 = 10$, but the average monthly income increases to \$4270.00, so that $Y_1 = 4.27$ and $\Delta Y = 0.27$, then the change in demand will be

$$\Delta q \approx \left.\frac{\partial q}{\partial Y}\right|_{\substack{p=10\\Y=4}} \cdot \Delta Y = 1 \cdot 0.27 = 0.27,$$

i.e., demand will *increase* by about 270 units/month.

(*) **Elasticity**

Just as we have generalized the *ordinary* derivatives of functions of one variable to *partial derivatives* of functions of several variables, so we can generalize the *elasticity* of a function $y = f(x)$ with respect to the variable x, to the *partial elasticities* of a function $w = f(x, y)$ of several variables with respect to each of its different variables.

We define the partial elasticities of w as we did for functions of one variable, and these definitions result in similar formulas to the ones we found in the one variable case.

Specifically, if only the variable x is changing (i.e., y is held fixed) so that

$$\Delta w = f(x + \Delta x, y) - f(x, y) \quad \text{and} \quad \%\Delta w = \frac{\Delta w}{w} \cdot 100\%$$

then the x-*elasticity of* w is given by

$$\eta_{w/x} = \lim_{\Delta x \to 0} \frac{\%\Delta w}{\%\Delta x} = \cdots = \frac{\partial w}{\partial x} \cdot \frac{x}{w}.$$

Likewise, if only the variable y is changing (so x is held fixed) we have

$$\Delta w = f(x, y + \Delta y) - f(x, y) \quad \text{and} \quad \%\Delta w = \frac{\Delta w}{w} \cdot 100\%$$

and the y-*elasticity of* w is given by

$$\eta_{w/y} = \lim_{\Delta y \to 0} \frac{\%\Delta w}{\%\Delta y} = \cdots = \frac{\partial w}{\partial y} \cdot \frac{y}{w}.$$

Moreover, all the basic uses of elasticity that we found in the one variable case continue to be valid ***with the added stipulation that only one variable is changing.***

Example. Returning to the demand function in the example above, $q = 3\ln(4Y^{1/2} - 0.5p)$, the *price-elasticity of demand*, $\eta_{q/p}$, when $p_0 = 10$ and $Y = 4$, is

$$\eta_{q/p}\Big|_{\substack{p=10\\Y=4}} = \left(\frac{\partial q}{\partial p} \cdot \frac{p}{q}\right)\Big|_{\substack{p=10\\Y=4}} \approx (-0.5) \cdot \frac{10}{3.296} \approx -1.517.$$

We can conclude that demand for the firm's product is *elastic* when $p = 10$ (and $Y = 4$), because $|\eta_{q/p}| > 1$. Relatively small (percentage) changes in price will result in relatively larger (percentage) changes in demand.

For example, a price increase from $p_0 = 10$ to $p_1 = 10.40$ is a

$$\frac{0.4}{10} \cdot 100\% = 4\%$$

change in price. This will result in a percentage change in demand of about

$$\%\Delta q \approx \left.\eta_{q/p}\right|_{\substack{p=10\\Y=4}} \cdot \%\Delta p \approx (-1.517) \cdot 4\% = -6.068\%.$$

I.e., a 4% increase in price results in a slightly more than 6% decrease in demand.

(*) **Quiz 5.** $z = \sqrt{2x + y^3} = \left(2x + y^3\right)^{1/2}$, find z_x and z_y.

Solution.

$$z_x = \frac{\cancel{1}}{\cancel{2}}\left(2x + y^3\right)^{-1/2} \cdot \cancel{2} = \left(2x + y^3\right)^{-1/2} \qquad \left(= \frac{1}{\sqrt{2x + y^3}}\right).$$

$$z_y = \frac{1}{2}\left(2x + y^3\right)^{-1/2} \cdot 3y^2 = \frac{3}{2}y^2\left(2x + y^3\right)^{-1/2} \qquad \left(= \frac{3y^2}{2\sqrt{2x + y^3}}\right).$$