(*) Quadratic Taylor polynomials in one variable.

The *quadratic* Taylor polynomial $T_2(t)$ for $f(t)$, centered at t_0 is given by

$$T_2(t) = f(t_0) + f'(t_0)(t - t_0) + \frac{f''(t_0)}{2}(t - t_0)^2,$$

which has the properties

- $T_1(t_0) = f(t_0)$,
- $T_1'(t_0) = f'(t_0)$ and
- $T_1''(t_0) = f''(t_0)$.

These key properties mean that $T_2(t)$ behaves very much like $f(t)$ at t_0, and as a consequence the approximation

$$f(t) \approx T_2(t)$$

is very accurate when t is close to t_0.

(*) Quadratic Taylor polynomials in two variables

Given a function $z = f(x, y)$ and a point (x_0, y_0), we we'll look for a *quadratic function in two variables* $T_2(x, y)$ that satisfies conditions analogous to those satisfied by $T_2(t)$ in the one-variable case.

Quadratic functions in two variables generally have the form

$$P(x, y) = \overbrace{\alpha}^{\text{constant}} + \overbrace{\beta x + \gamma y}^{\text{linear terms}} + \overbrace{\delta x^2 + \varepsilon xy + \phi y^2}^{\text{quadratic terms}}.$$

For the purposes of finding the quadratic Taylor polynomial $T_2(x, y)$ for the function $f(x, y)$, centered at the point (x_0, y_0), it is more convenient to write it as

$$\begin{aligned} T_2(x, y) = A &+ B(x - x_0) + C(y - y_0) \\ &+ D(x - x_0)^2 + E(x - x_0)(y - y_0) + F(y - y_0)^2, \end{aligned}$$

because it makes finding the values of the coefficients $(A, \ldots, F)$ *much* easier. The conditions that we want T_2 to satisfy are:

- $T_2(x_0, y_0) = f(x_0, y_0)$
- $T_{2x}(x_0, y_0) = f_x(x_0, y_0)$
- $T_{2y}(x_0, y_0) = f_y(x_0, y_0)$
- $T_{2xx}(x_0, y_0) = f_{xx}(x_0, y_0)$
- $T_{2xy}(x_0, y_0) = f_{xy}(x_0, y_0)$
- $T_{2yy}(x_0, y_0) = f_{yy}(x_0, y_0)$

These conditions lead to the following equations for the coefficients of $T_2(x, y)$...

Constant coefficient:

$$f(x_0, y_0) = T_2(x_0, y_0) = A + 0 + 0 + 0 + 0 + 0 = A \implies A = f(x_0, y_0).$$

Linear coefficients:

$$T_{2x} = B + 2D(x - x_0) + E(y - y_0)$$

and

$$T_{2y} = C + E(x - x_0) + 2F(y - y_0),$$

so

$$f_x(x_0, y_0) = T_{2x}(x_0, y_0) = B + 0 + 0 \implies B = f_x(x_0, y_0)$$

$$f_y(x_0, y_0) = T_{2y}(x_0, y_0) = C + 0 + 0 \implies C = f_y(x_0, y_0)$$

Quadratic terms:

$$T_{2xx} = 2D, \quad T_{2xy} = E \quad \text{and} \quad T_{2yy} = 2F,$$

so

$$f_{xx}(x_0, y_0) = T_{2xx}(x_0, y_0) = 2D \implies D = \frac{f_{xx}(x_0, y_0)}{2}$$

$$f_{xy}(x_0, y_0) = T_{2xy}(x_0, y_0) = E \implies E = f_{xy}(x_0, y_0)$$

$$f_{yy}(x_0, y_0) = T_{2yy}(x_0, y_0) = 2F \implies F = \frac{f_{yy}(x_0, y_0)}{2}$$

Conclusion:

The quadratic Taylor polynomial for $f(x, y)$, centered at (x_0, y_0) is

$$\begin{aligned} T_2(x, y) = {} & f(x_0,y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ & + \frac{f_{xx}(x_0, y_0)}{2}(x - x_0)^2 + f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + \frac{f_{yy}(x_0, y_0)}{2}(y - y_0)^2 \end{aligned}$$

Quadratic approximation: If (x, y) is close to (x_0, y_0), then

$$f(x, y) \approx T_2(x, y).$$

Comment: '(x, y) is close to (x_0, y_0)' means that the distance in the plane between these two points is small. If both $|x - x_0|$ is small and $|y - y_0|$ is small, then the distance in the plane between (x, y) and (x_0, y_0),

$$\sqrt{(x - x_0)^2 + (y - y_0)^2},$$

will also be small.

(*) **Example**

Find the quadratic Taylor polynomial for

$$f(x,y) = (2x+5y)^{1/2}$$

centered at the point $(x_0, y_0) = (3,2)$.

This amounts to (a) finding the coefficients and (b) putting the pieces together correctly. To find the coefficients, we need to find the first and second order partial derivatives of $f(x,y)$:

$$f_x = (2x+5y)^{-1/2},\ f_y = \frac{5}{2}(2x+5y)^{-1/2},\ f_{xx} = -(2x+5y)^{-3/2},$$

$$f_{xy} = -\frac{5}{2}(2x+5y)^{-3/2} \quad \text{and} \quad f_{yy} = -\frac{25}{4}(2x+5y)^{-3/2}.$$

Therefore

$$f(3,2) = 16^{1/2} = 4, \quad f_x(3,2) = 16^{-1/2} = \frac{1}{4}, \quad f_y(3,2) = \frac{5}{2}\cdot 16^{-1/2} = \frac{5}{8}$$

$$f_{xx}(3,2) = -16^{-3/2} = -\frac{1}{64}, \quad f_{xy}(3,2) = -\frac{5}{2}\cdot 16^{-3/2} = -\frac{5}{128}$$

$$\text{and} \quad f_{yy}(3,2) = -\frac{25}{4}\cdot 16^{-3/2} = -\frac{25}{256}.$$

So

$$T_2(x,y) = 4 + \frac{1}{4}(x-3) + \frac{5}{8}(y-2) - \frac{1}{128}(x-3)^2 - \frac{5}{128}(x-3)(y-2) - \frac{25}{512}(y-2)^2$$

We can use this to approximate

$$\sqrt{17} = f(3.25, 2.1)\ldots$$

$$\begin{aligned} f(3.25, 2.1) &= 4 + \frac{1}{4}\cdot\frac{1}{4} + \frac{5}{8}\cdot\frac{1}{10} - \frac{1}{128}\cdot\frac{1}{16} - \frac{5}{128}\cdot\frac{1}{4}\cdot\frac{1}{10} - \frac{25}{512}\cdot\frac{1}{100} \\ &= 4 + \frac{1}{16} + \frac{1}{16} - \frac{1}{2048} - \frac{1}{1024} - \frac{1}{2048} \\ &= \frac{8192+128+128-1-2-1}{2048} \\ &= \frac{2111}{512} \end{aligned}$$

Quadratic approximation: $\sqrt{17} \approx \frac{2111}{512} = 4.123046875$

Calculator: $\sqrt{17} = 4.1231056\ldots$

Error: $\left|\sqrt{17} - \frac{2111}{512}\right| < 0.00006.$

(*) **Looking ahead...**

(*) We won't be computing specific quadratic Taylor polynomials for approximation (or any other) purposes.

(*) We ***will*** be using the quadratic Taylor polynomial in two variables to understand the *second derivative test* in two variables. To this end, the key observation is the following.

If $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ (we say that (x_0, y_0) is a *critical point* in this case), then the quadratic Taylor polynomial for $f(x, y)$ centered at (x_0, y_0) is

$$T_2(x,y) = f(x_0,y_0) + \frac{f_{xx}(x_0,y_0)}{2}(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{f_{yy}(x_0,y_0)}{2}(y-y_0)^2$$

It follows in this case that if (x, y) is close to (x_0, y_0), then

$$\begin{aligned} f(x,y) - f(x_0,y_0) &\approx T_2(x,y) - f(x_0,y_0) \\ &= \frac{f_{xx}(x_0,y_0)}{2}(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \frac{f_{yy}(x_0,y_0)}{2}(y-y_0)^2 \end{aligned}$$

This observation is the foundation of the second derivative test in two variables.

(*) **Quiz 6**

$z = x^2 \ln(2x + 3y)$. Find z_{xx} and z_{xy}.

Solution:

$z_x = 2x\ln(2x+3y) + x^2 \cdot \dfrac{2}{2x+3y} = 2x\ln(2x+3y) + \dfrac{2x^2}{2x+3y}$ (product rule and chain rule).

So...

$$\begin{aligned} z_{xx} = \left(2x\ln(2x+3y) + \frac{2x^2}{2x+3y}\right)_x &= 2\ln(2x+3y) + 2x\cdot\frac{2}{2x+3y} + \frac{4x(2x+3y) - 2(2x^2)}{(2x+3y)^2} \\ &= 2\ln(2x+3y) + \frac{4x}{2x+3y} + \frac{4x^2+12xy}{(2x+3y)^2} \\ &\left(= 2\ln(2x+3y) + \frac{12x(x+2y)}{(2x+3y)^2}\right) \end{aligned}$$

and

$$\begin{aligned} z_{xy} = \left(2x\ln(2x+3y) + \frac{2x^2}{2x+3y}\right)_y &= 2x\cdot\frac{3}{2x+3y} + \frac{0 - 3(2x^2)}{(2x+3y)^2} \\ &= \frac{6x}{2x+3y} - \frac{6x^2}{(2x+3y)^2} \qquad \left(= \frac{6x(x+3y)}{(2x+3y)^2}\right) \end{aligned}$$