(*) Substitution

From the chain rule (for the differentiation of composite functions) we know that

$$\frac{d}{dx}(F(g(x)) = F'(g(x))g'(x).$$

Writing $F'(x) = f(x)$, this gives the corresponding rule for integration

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

The difficulty lies in *recognizing* that an integrand is the derivative of a composite function. What we do is look for the following pattern: an integrand that is the product of a composite function and a simpler function, i.e., an integral that looks like this

$$\int f(g(x))k(x)\,dx.$$

If $k(x) = g'(x)$, then we're in business. To compute $\int f(g(x))g'(x)\,dx$, we *substitute* (rename) $u = g(x)$, which means that $du = g'(x)\,dx$ so that

$$\int f(g(x))g'(x)\,dx = \int f(\overbrace{g(x)}^{u})\overbrace{g'(x)dx}^{du} = \int f(u)\,du.$$

We still have some work to do, but $\int f(u)du$ is a simpler integral than the one we started with.

Example. To compute $\int 2x\sqrt{x^2+3}\,dx$, we first notice that $(x^2+3)' = 2x$, so that substituting $u = x^2+3$ means that $du = 2x\,dx$. This allows us to simplify (and calculate) the integral:

$$\int 2x\sqrt{x^2+3}\,dx = \int \underbrace{\sqrt{x^2+3}}_{\sqrt{u}}\overbrace{2x\,dx}^{du} = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}(x^2+3)^{3/2} + C.$$

The last step is to replace u by x^2+3 in the answer.

(*) Adjusting the constant factor

More generally, if in the integral $\int f(g(x))k(x)\,dx$, $k(x) = a \cdot g'(x)$ for some (nonzero) constant a, then substituting $u = g(x)$ will still work. In this case we have $du = g'(x)\,dx$ and since $k(x) = ag'(x)$, it follows that $k(x)\,dx = ag'(x)\,dx = a(g'(x)\,dx) = a\,du$, so

$$\int f(g(x))k(x)\,dx = \int f(\overbrace{g(x)}^{u})\overbrace{ag'(x)\,dx}^{a\,du} = \int f(u)\,a\,du = a\int f(u)\,du.$$

Example. To compute $\int 5x\sqrt{x^2+3}\,dx$, we first notice that $(x^2+3)' = 2x$, so that substituting $u = x^2+3$ means that $du = 2x\,dx$. Dividing by 2 shows that $x\,dx = \frac{1}{2}\,du$ and multiplying by 5 results in the identity $5x\,dx = \frac{5}{2}\,du$. Now we can integrate:

$$\int 5x\sqrt{x^2+3}\,dx = \int \underbrace{\sqrt{x^2+3}}_{\sqrt{u}}\overbrace{5x\,dx}^{\frac{5}{2}du} = \frac{5}{2}\int u^{1/2}\,du = \frac{5}{2}\cdot\frac{u^{3/2}}{3/2} + C = \frac{5}{3}(x^2+3)^{3/2} + C.$$

(*) **Quiz 1.** *Find the function $y = f(x)$ satisfying (i) $y' = \dfrac{x^2+2x+3}{5x}$, and (ii) $y(1) = 3$.*

Solution.

Step 1. *Integrate*

$$\int \frac{x^2+2x+3}{5x}dx = \int \frac{1}{5}x + \frac{2}{5} + \frac{3}{5}\cdot\frac{1}{x}\,dx = \frac{x^2}{10} + \frac{2}{5}x + \frac{3}{5}\ln|x| + C,$$

i.e., the solution y has the form $y = \frac{1}{10}x^2 + \frac{2}{5}x + \frac{3}{5}\ln|x| + C$, for some (as of yet undetermined) constant C.

Step 2. *Solve for C*

From the given data $y(1) = 3$, we see that

$$3 = y(1) = \frac{1}{10}\cdot 1^2 + \frac{2}{5}\cdot 1 + \frac{3}{5}\ln|1| + C = \frac{1}{10} + \frac{2}{5} + \frac{3}{5}\cdot 0 + C = \frac{1}{2} + C \implies C = 3 - \frac{1}{2} = \frac{5}{2},$$

so the function is

$$y = \frac{1}{10}x^2 + \frac{2}{5}x + \frac{3}{5}\ln|x| + \frac{5}{2}.$$