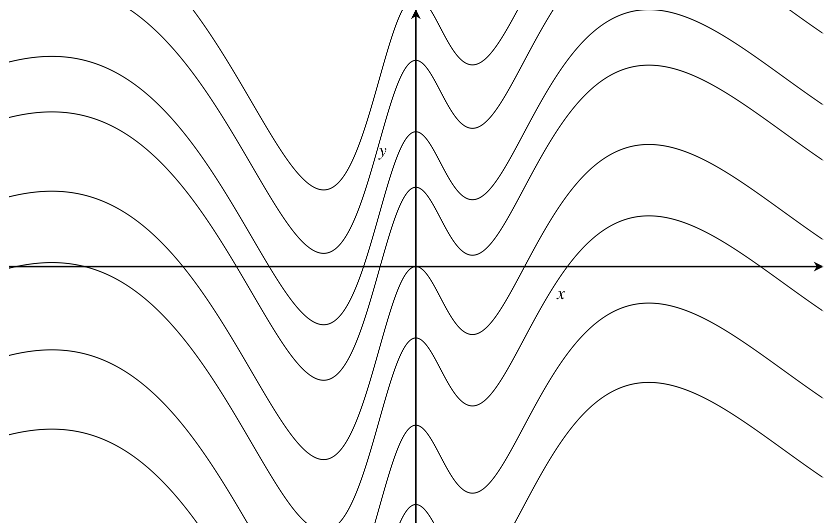
(*) Theoretical mumbo jumbo (and a pretty picture)

Given a continuous function $f(x)$, there are infinitely many functions whose derivative is equal to $f(x)$. If $F(x)$ is one of these, then all the others have the form $F(x)+C$ for some unspecified (indefinite) value of the constant C. We express this with the notation

$$\int f(x)\,dx = F(x) + C.$$

Since all of the antiderivatives of $f(x)$ differ from each other by an additive constant, their graphs are all *parallel*, which means, among other things, that if $G'(x) = H'(x) = f(x)$, then either $G(x) = H(x)$ for all x,[†] or $G(x) \neq H(x)$ for *any* x. I.e., graphs of different antiderivatives of $f(x)$ *do not intersect*, as in the image below.



Hypothetically, if we sketch all of the antiderivatives of $f(x)$, then the entire plane would be covered. Combining this observation with the fact that the graphs of different antiderivatives don't intersect leads to the following useful fact:

If x_0 is in the interval (a,b), $f(x)$ is continuous in this interval and y_0 is any value, then there exists a unique function $F(x)$ satisfying (i) $F'(x) = f(x)$ (in (a,b)) and (ii) $F(x_0) = y_0$.

(*) Example

Find the function $y = f(x)$ satisfying (i) $y' = x^2 - 3x + 1$ and (ii) $y(1) = -1$.

Step 1. *Integrate*

$$\int x^2 - 3x + 1\,dx = \frac{x^3}{3} - 3\frac{x^2}{2} + x + C.$$

This means that solution has the form $y = \dfrac{x^3}{3} - 3\dfrac{x^2}{2} + x + C$, and it remains to find the unique value of C that works.

[†]For all x where $f(x)$ is defined.

Step 2. *Solve for C*

Use the given initial value $y(1) = -1$:

$$-1 = y(1) = \frac{1^3}{3} - 3\frac{1^2}{2} + 1 + C = -\frac{1}{6} + C \implies C = -\frac{5}{6}.$$

Step 3. Solution: $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + x - \frac{5}{6}$.