## (*) Separable differential equations

In its most general form, a (first order, ordinary) differential equation is an equation of the form

$$\Phi(y, y', x) = 0,$$

whose solutions are functions $y = f(x)$. In general these equations can be very difficult to solve explicitly and frequently people use computers and numerical algorithms to generate approximate solutions.

On the other hand, certain differential equations can be relatively easy to solve, among these are ***separable*** differential equations. These are differential equations which can be put in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}.$$

This type of equation is called *separable* because multiplying the equation by $h(y)\cdot dx$ results in an equation where the variables $x$ and $y$ have been *separated*, an expression that equates two *differentials*,

$$h(y)\,dy = g(x)\,dx,$$

(hence the name *differential equation*). Integrating both sides of the separated equation leads to an *algebraic* equation,

$$\int h(y)\,dy = \int g(x)\,dx \qquad \Longrightarrow \qquad H(y) = G(x) + C,$$

which is called an *implicit solution* of the original differential equation (because it implies a relation between $y$ and $x$). Solving the algebraic equation for $y$ results in an explicit solution,

$$y = H^{-1}(G(x) + C),$$

or more accurately, a family of solutions, one for each possible value of $C$. Finally, given *data* in the form $y(x_0) = y_0$, we can solve for $C$ and find the unique solution $y = f(x)$ of the *initial value problem*

$$y' = \frac{g(x)}{h(y)} \quad \text{and} \quad y(x_0) = y_0.$$

(It can be shown that if $g(x)$ is continuous in an interval around $x_0$ and $h(y)$ is continuous and $h(y) \neq 0$ in an interval around $y_0$, then a unique solution does exist. Finding it easily is another question.)

**Example 1.** Find the function $y = f(x)$ satisfying

$$y' = \frac{2x+1}{y+2} \quad \text{and} \quad y(1) = 1.$$

**Step 1.** Separate:

$$y' = \frac{2x+1}{y+2} \Longrightarrow \frac{dy}{dx} = \frac{2x+1}{y+2} \Longrightarrow y + 2\,dy = 2x + 1\,dx.$$

**Step 2.** Integrate:

$$\int y + 2\,dy = \int 2x + 1\,dx \implies \frac{y^2}{2} + 2y = x^2 + x + C.$$

**Step 3.** Solve for $y$:

$$\frac{y^2}{2} + 2y = x^2 + x + C \implies y^2 + 4y = 2x^2 + 2x + C \implies y^2 + 4y - \left(2x^2 + 2x + C\right) = 0$$

The last equation on the right is a quadratic equation in $y$ of the from $ay^2 + by + c = 0$, where $a = 1$, $b = 4$ and $c = -\left(2x^2 + 2x + C\right)$, and we can solve it using the quadratic formula:

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 + 4\left(2x^2 + 2x + C\right)}}{2}$$

$$= \frac{-\cancel{4}^{\,2} \pm \cancel{2}\sqrt{4 + 2x^2 + 2x + C}}{\cancel{2}} \quad \text{(factoring 4 out of the } \sqrt{\phantom{x}}\text{)}$$

$$= -2 \pm \sqrt{2x^2 + 2x + C} \quad \text{(‘absorbing’ the term 4 into the constant } C\text{)}$$

So $y = -2 \pm \sqrt{2x^2 + 2x + C}$ and to solve for $C$ and determine whether the ‘$\pm$’ is $+$ or $-$, we use the data, $y(1) = 1$.

First, since $y(1) = 1 > 0$, we must choose the $+$ sign, otherwise $y < 0$ when $x = 1$. This means that

$$1 = y(1) = -2 + \sqrt{2 \cdot 1^2 + 2 \cdot 1 + C} \implies \sqrt{4 + C} = 3 \implies C = 5,$$

so the (unique) solution to this initial value problem is

$$y = \sqrt{2x^2 + 2x + 5} - 2.$$

**Exercise:** Compute $y'$ and verify that it satisfies the differential equation.

(*) **Elasticity**

Given a functional relation $y = f(x)$, the $x$-elasticity of $y$ is defined as

$$\eta_{y/x} = \lim_{\Delta x \to 0} \frac{\%\Delta y}{\%\Delta x} = \frac{dy}{dx} \cdot \frac{x}{y},$$

where $\%\Delta y$ and $\%\Delta x$ are the percentage-changes in $y$ and $x$, respectively.

Economic theory will sometimes dictate that the $x$-elasticity of $y$ has certain characteristics, and these characteristics can lead to a differential equation for the (unknown) function $y = f(x)$.

**Example 2.** A short term production function

$$Q = P(L)$$

is one where the capital input ($K$) is considered to be fixed while the labor input ($L$) is variable. The variable $Q$ is output.

In some contexts, economists will specify that the *labor-elasticity of output* is constant. The condition $\eta_{Q/L} = \beta$, where $\beta$ is the (typically unknown) constant value of of the elasticity, leads to the *separable* differential equation

$$\frac{dQ}{dL} \cdot \frac{L}{Q} = \beta \quad \Longrightarrow \quad \frac{dQ}{Q} = \beta \cdot \frac{dL}{L}.$$

Integrating both sides

$$\int \frac{dQ}{Q} = \beta \int \frac{dL}{L},$$

leads to the implicit relation

$$\ln Q = \beta \ln L + C.$$

To solve for $Q$, we exponentiate both sides,

$$e^{\ln Q} = e^{\beta \ln L + C} = e^C \cdot e^{\ln(L^\beta)} \quad \Longrightarrow \quad Q = AL^\beta,$$

where $A = e^C$. In other words, if the labor elasticity of output is constant then the short term production function is a *power function*, where the power is equal to the constant elasticity.[†]

This example illustrates another common feature of differential equations, namely that they might include unspecified parameters. The specific solution of such an equation might require more than one data point.

**Example 2.** (continued) Suppose that a short term production function $Q = P(L)$ has constant labor-elasticity of output, then we know that

$$Q = AL^\beta$$

but we can't know the values of $A$ or $\beta$ without more information. Since there are two unknown parameters, it stands to reason that we will need two data points, so suppose that when labor input is $L_0 = 100$, the output is $Q_0 = 1000$ and when labor input is $L_1 = 200$, the output is $Q_1 = 1500$. This data, together with the relation $Q = AL^\beta$ leads to the pair of equations below for $A$ and $\beta$:

$$1000 = A(100)^\beta \tag{1}$$
$$1500 = A(200)^\beta \tag{2}$$

To solve this pair of equations, we can (i) take logarithms of both sides, which converts them into the following pair of linear equations (for $\beta$ and $\ln A$)

$$\ln 1000 = \ln A + \beta \ln 100$$
$$\ln 1500 = \ln A + \beta \ln 200,$$

[†]The conclusion is of course general — if $\eta_{y/x} = \beta$, then $y = Ax^\beta$, where $A$ is another constant — it doesn't matter what the variables $x$ and $y$ represent.

which we know how to solve,[‡] or we can (ii) proceed as follows.

The quotient of the lefthand side of (2) by the lefthand side of (2) is equal to the quotient of the righthand side of (2) by the righthand side of (2):

$$\frac{1500}{1000} = \frac{\cancel{A}(200)^\beta}{\cancel{A}(100)^\beta} \implies 1.5 = 2^\beta,$$

This has the effect of eliminating the parameter $A$. Now, taking logarithms of both sides of the equation on the right and moving things around, gives the value of $\beta$:

$$\ln 1.5 = \beta \ln 2 \implies \beta = \frac{\ln 1.5}{\ln 2} \quad (\approx 0.585).$$

Finally, returning to (1) again, we can solve for $A$:

$$1000 = A(100)^\beta \implies A = 1000 \cdot (100)^{-\beta} = 1000 \cdot (100)^{-\ln 1.5/\ln 2} \approx 67.62.$$

So the production function is

$$Q \approx 67.22 L^{0.585}.$$

**Exercise.** Solve the pair of linear equations at the bottom of the last page and check that you obtain the same values for $A$ and $\beta$ as above.

In many cases, elasticity is not constant. This is certainly common for price-elasticity of demand, where the elasticity changes as the price changes, i.e., where the price-elasticity of demand is a function of the price. Happily,[§] these assumptions also lead to separable differential equations

**Example 3.** The price-elasticity of demand $q$ for a certain good is assumed to be *proportional to* the square root of the price $p$ of that good. When the price is $p_0 = 9$, the demand is $q_0 = 500$ and when the price is $p_1 = 25$, the demand is $q_1 = 300$.

What will demand be when the price is $p_2 = 36$?

The phrase *'proportional to'* means *'multiple of'*, so that the assumption above about the price-elasticity of demand leads to the separable differential equation

$$\eta_{q/p} = k\sqrt{p} \implies \frac{dq}{dp} \cdot \frac{p}{q} = k\sqrt{p} \implies \frac{dq}{q} = k\frac{\sqrt{p}}{p}\, dp = kp^{-1/2}\, dp,$$

where $k$ is the (unknown) constant of proportionality. Integrating both sides of the equation on the right yields an implicit relation between $q$ and $p$,

$$\int \frac{dq}{q} = k \in p^{-1/2}\, dp \implies \ln q = k \cdot \frac{p^{1/2}}{1/2} + C = k_1 p^{1/2} + C,$$

where $k_1 = 2k$ is still an unknown constant. As in the previous example, we solve for $q$ by exponentiating both sides of the last equation,

$$e^{\ln q} = e^{k_1 p^{1/2} + C} = e^C \cdot e^{k_1 p^{1/2}} \implies q = Ae^{k_1 p^{1/2}},$$

[‡]Right?

[§]Depending on your perspective.

where (also as before) $A = e^C$.

Once again, we now use the data to solve for $A$ and $k_1$, using the two equations $q_0 = Ae^{k_1\sqrt{p_0}}$ and $q_1 = Ae^{k_1\sqrt{p_1}}$:

$$500 = Ae^{k_1\sqrt{9}} = Ae^{3k_1}$$
$$300 = Ae^{k_1\sqrt{25}} = Ae^{5k_1}$$

Dividing left side by left side and right side by right side again eliminates the $A$ (again),

$$\frac{300}{500} = \frac{\not{A}e^{5k_1}}{\not{A}e^{3k_1}} \implies 0.6 = e^{5k_1-3k_1} = e^{2k_1},$$

and taking logarithms again gives

$$\ln 0.6 = 2k_1 \implies k_1 = \frac{\ln 0.6}{2} \quad (\approx -0.2554).$$

Next, using the first data point (or the second) we find that

$$500 = Ae^{3k_1} \implies A = 500e^{-3k_1} \quad (\approx 1075.83),$$

and finally, when the price is $p_2 = 36$, the demand will be

$$q_2 = Ae^{k_1\sqrt{36}} = 500e^{-3k_1}e^{6k_1} = 500e^{3k_1} = 500e^{\frac{3}{2}\ln 0.6} \approx 232.38.$$

(*) **Population growth models — Exponential growth**

The simplest model for population growth is based on the assumption that the population grows *at a rate proportional to its size*. This assumption considers only the factors intrinsic to the population itself, e.g., birthrate, and leads to the differential equation

$$\frac{dP}{dt} = rP,$$

where $P(t)$ is the size of the population at time $t$, and $r$ is the *intrinsic growth rate*. This equation is easy to solve (after separating the variables):

$$\frac{dP}{P} = r\,dt \implies \int \frac{dP}{P} = r\int dt \implies \ln P = rt + C \implies P = Ae^{rt},$$

where

- $P > 0$, so we can drop the absolute value sign.
- $A = e^C$, and in fact...
- $A = P(0) = P_0$, the initial population size.

I.e., the *exponential growth* model is

$$P(t) = P_0e^{rt}.$$

**Example 5.** The population of a small island in the year 1950 was 870 people, and in the year 2000, the population was 1250. Assuming exponential growth, what will the island's population be in the year 2050? How about in 2150?

Based on the assumption of exponential growth, we have

$$P(t) = 870e^{rt},$$

with time being measured in years, and $t = 0$ corresponding to the year 1950. This means that

$$\begin{aligned} 1250 = P(50) = 870e^{50r} &\implies e^{50r} = \frac{1250}{870} \\ &\implies 50r = \ln(125/87) \\ &\implies r = \frac{1}{50}\ln(125/87) \quad (\approx 0.00725) \end{aligned}$$

Therefore

$$P(100) = 870e^{100r} \approx 1796 \quad \text{and} \quad P(200) = 870e^{200r} \approx 3708.$$

The exponential growth model $P = P_0e^{rt}$ can be quite accurate in the short run, but not in the long run, because an exponentially growing population will eventually outstrip its resources. This observation leads to a different model.

(*) **Population growth models — Logistic growth**

This model accounts for the fact that populations grow in environments that have limited resources. Such an environment has a ***carrying capacity***, which is the maximum (sustainable) size for the population growing there.

The logistic model is based on the following assumptions/requirements.

(i) When the population is small *relative to the carrying capacity*, it should grow at a rate (approximately) proportional to its size (like exponential growth).

(ii) As the population gets close to the carrying capacity in size, the growth rate should approach 0.

(iii) If the initial population size is bigger than the carrying capacity, the growth rate should be negative.

(iv) The model should be as simple as possible.

If the carrying capacity is $M$ and the intrinsic growth rate is $r$, then the first three assumptions translate to

(i) If $P/M \approx 0$, then $\dfrac{dP}{dt} \approx rP$.

(ii) If $P/M \approx 1$, then $\dfrac{dP}{dt} \approx 0$.

(iii) If $P/M > 1$, then $\dfrac{dP}{dt} < 0$.

These assumptions (and the desire for as simple a model as possible), lead to the *logistic equation*:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right),$$

which satisfies all three conditions:

(*) If $P/M \approx 0$, then $rP\left(1 - \frac{P}{M}\right) \approx rP(1-0) = rP$

(*) If $P/M \approx 1$, then $rP\left(1 - \frac{P}{M}\right) \approx rP(1-1) = 0$

(*) If $P/M > 1$, then $rP\left(1 - \frac{P}{M}\right) < 0$

The logistic equation is separable and is solved as follows.

First, factor out $1/M$ from the second factor on the right

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right) = \frac{r}{M}P\left(M - P\right).$$

Then separate

$$\frac{dP}{P(M-P)} = \frac{r}{M}\,dt.$$

Then integrate (using formula #5 in the appendix, with $a = M$ and $b = -1$)

$$\int \frac{dP}{P(M-P)} = \int \frac{r}{M}\,dt \implies \frac{1}{M}\ln\left|\frac{P}{M-P}\right| = \frac{rt}{M} + C.$$

Finally, solve for $P$

$$\frac{1}{\cancel{M}}\ln\left|\frac{P}{M-P}\right| = \frac{rt}{\cancel{M}} + C \implies \ln\left|\frac{P}{M-P}\right| = rt + C$$
$$\implies \frac{P}{M-P} = Ae^{rt}$$

where $A = \pm e^C$.

A little more algebra:

$$P = (M-P)Ae^{rt} = AMe^{rt} - APe^{rt} \implies P + APe^{rt} = AMe^{rt}$$
$$\implies P(1 + Ae^{rt}) = AMe^{rt}$$
$$\implies P = \frac{AMe^{rt}}{1 + Ae^{rt}}$$

The formula for $P(t)$ can be further manipulated in different ways.

One approach is to divide the numerator and denominator by $Ae^{rt}$ which gives

$$P = \frac{M}{1 + be^{-rt}},$$

where $b = A^{-1}$. (Our textbook does it this way.)

Another approach is to replace $A$ by a more meaningful parameter. Both $M$ and $r$ have meaningful interpretations, and it is relatively easy to express $A$ in terms of $M$ and the *initial population size* $P_0$.

If $t = 0$, then

$$P_0 = P(0) = \frac{AM}{1+A} \implies AM = P_0(1+A) = P_0 + AP_0$$
$$\implies AM - AP_0 = P_0 \implies A(M - P_0) = P_0$$
$$\implies A = \frac{P_0}{M - P_0}$$

Now, substitute this for $A$ in the first expression for $P$

$$P = \frac{AMe^{rt}}{1 + Ae^{rt}} \implies \frac{\frac{P_0}{M-P_0}Me^{rt}}{1 + \frac{P_0}{M-P_0}e^{rt}}$$

Finally, multiply both top and bottom by $(M - P_0)e^{-rt}$, which gives

$$P(t) = \frac{P_0 M}{P_0 + (M - P_0)e^{-rt}}.$$

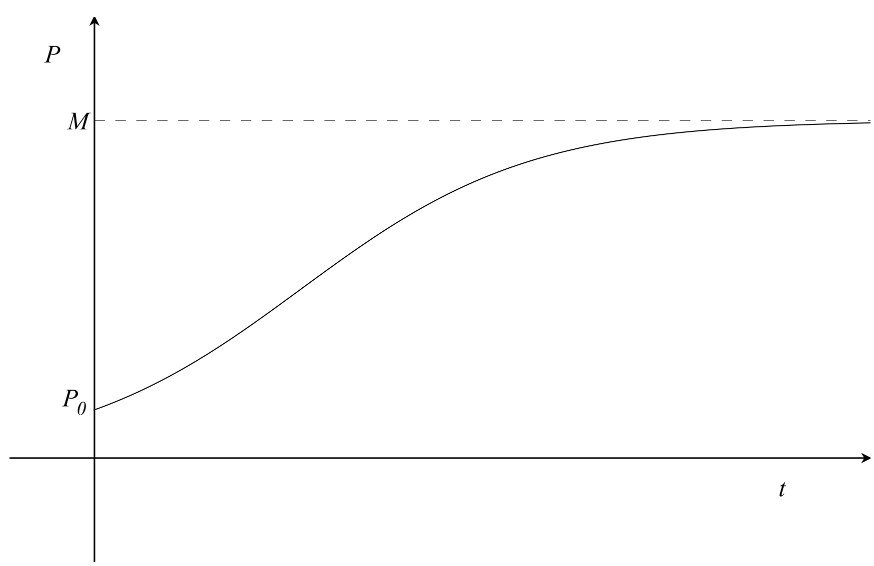


Figure 1: Graph of $P = \dfrac{P_0 M}{P_0 + (M - P_0)e^{-rt}}$

**Example.** A new virus is spreading on a closed network of 5000 computers. By the time the virus is first spotted, 25 computers are infected, and two hours later 200 computers are infected. Assuming logistic growth, how many hours before half the network is infected?

In this example, we know the carrying capacity $M = 5000$ and the initial population size $P_0 = 25$, so the number of infected computers at time $t$ is

$$P(t) = \frac{25 \cdot 5000}{25 + 4975e^{-rt}} = \frac{5000}{1 + 199e^{-rt}}.$$

From the data, we have

$$P(2) = \frac{5000}{1+199e^{-2r}} = 200 \implies 5000 = 200(1+199e^{-2r})$$

$$\implies 25 = 1 + 199e^{-2r}$$

$$\implies 24 = 199e^{-2r}$$

$$\implies e^{2r} = \frac{199}{24}$$

$$\implies r = \frac{1}{2}\ln(199/24)$$

Finally, solve the equation $P(t_1) = 5000/2 = 2500$:

$$2500 = \frac{5000}{1+199e^{-rt_1}} \implies 1 + 199e^{-rt_1} = \frac{5000}{2500} = 2$$

$$\implies 199e^{-rt_1} = 1 \implies e^{rt_1} = 199$$

$$\implies t_1 = \frac{\ln 199}{r} = \frac{\ln 199}{\frac{1}{2}\ln(199/24)} \approx 5$$

**Conclusion:** Half the network will be infected about 5 hours after the virus is first detected.