(*) The Gini Coefficient

The *Lorenz curve*, $y = f(x)$, for a nation's economy describes the income (or wealth) distribution of that nation. For $0 \le x \le 1$, $y = f(x)$ gives the proportion of the national income earned by the lowest-earning $x \times 100\%$ of the population. For example, if the lowest-earning 10% of the population earn 1% of the national income, then $f(0.1) = 0.01$ and if the lowest-earning 50% of the population earn 22% of the national income, then $f(0.5) = 0.22$.

The Lorenz curve has the following characteristics.

- $f(0) = 0$ and $f(1) = 1$, because 0% of the population earns 0% of the income and 100% of the population earns 100% of the income.
- $f(x)$ is increasing because the bigger the proportion of the population, the more they earn.
- Lorenz curves are *concave up*, i.e., their derivatives (assuming that they are differentiable) are *increasing*.

Thus, the typical Lorenz curve looks like the (red) one in Figure 1 below.

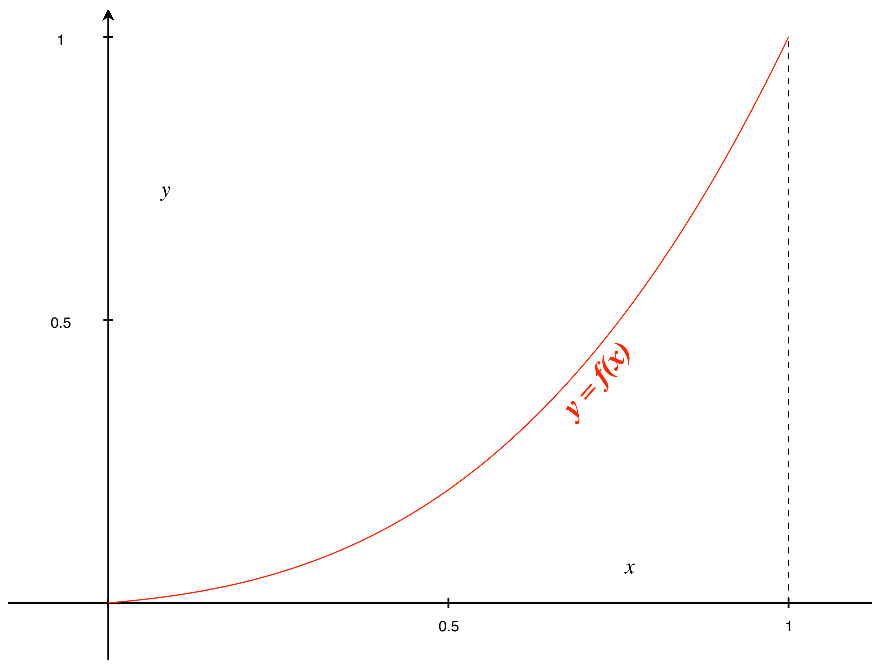


Figure 1: A typical Lorenz curve.

In the nation whose income distribution is described by the curve above, the lower-income earners earn less than the higher-income earners — i.e., there is *inequality* in the income distribution. In a (hypothetical) nation in which income is equally distributed among the entire population, the first 1% of the population earns 1% of the income, the next 1% of the population also earns 1% of the income, so the first 2% of the population earns 2% of the income, etc. For such a nation the Lorenz curve would be given by $y = x$, so that for each x, the lowest $x \times 100\%$ of the population earns $x \times 100\%$ of the income.

Comparing the Lorenz curve $y = f(x)$ of a nation to the curve of perfect equality $y = x$, we see that the more unequal the income distribution, the bigger the area of the region trapped between the curves $y = x$ and the Lorenz curve. This is depicted in Figure 2 below, where the Lorenz curve $y = g(x)$ describes a nation where income is more unequally distributed than the nation with Lorenz curve $y = f(x)$.

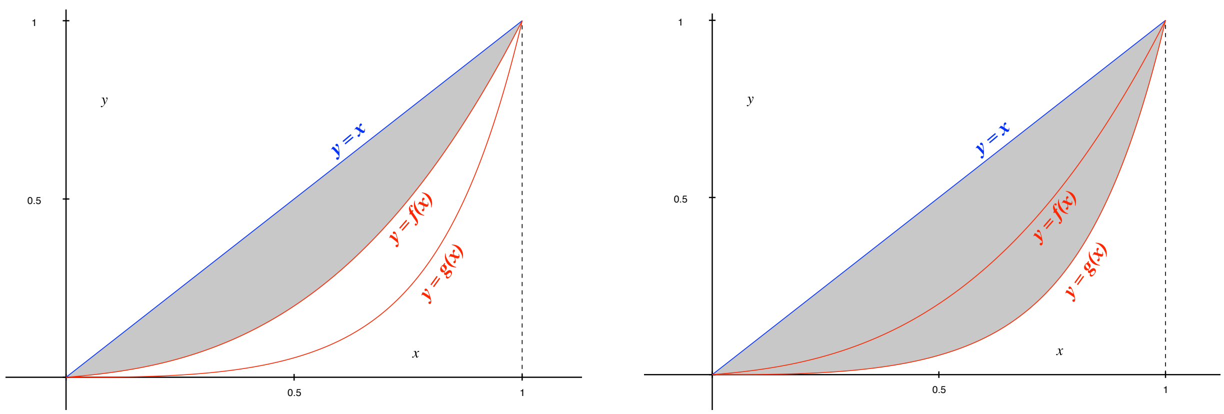
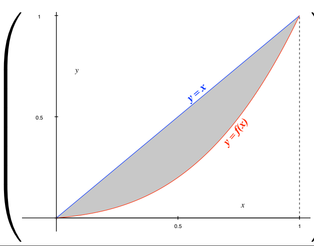
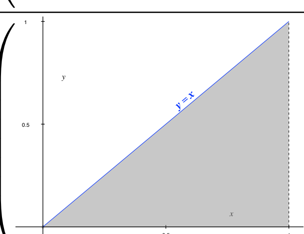


Figure 2: Two Lorenz curves compared to the curve of perfect equality.

The Italian sociologist Corrado Gini suggested measuring the inequality for a given nation by the ratio of two areas: (i) the area of the region between $y = x$ and $y = f(x)$ and (ii) the area of the triangle between $y = x$ and the interval $[0, 1]$ on the x-axis. This ratio is called the *Gini coefficient of inequality*, γ. I.e.,

$\gamma =$ area() / area() $= \dfrac{\int_0^1 x - f(x)\,dx}{\frac{1}{2}} = 2\int_0^1 x - f(x)\,dx.$

Example. Find the Gini coefficient of inequality for the nation with Lorenz curve given by $f(x) = 2^x - 1$.

$$\gamma = 2\int_0^1 x - (2^x - 1)\,dx = 2\int_0^1 x + 1 - 2^x\,dx = 2\left(\frac{x^2}{2} + x - \frac{2^x}{\ln 2}\Big|_0^1\right) = 3 - \frac{2}{\ln 2} \approx 0.1146.$$

Quiz 3. *Compute the (definite) integral* $\int_0^4 \sqrt{4x+9}\,dx$.

Solution: Make the substitution $u = 4x+9$, which entails $du = 4\,dx$ so $dx = \frac{1}{4}\,du$. Also the limits of integration change: $x = 0 \implies u = 4\cdot 0+9 = 9$ and $x = 4 \implies u = 4\cdot 4+9 = 25$. So...

$$\int_0^4 \sqrt{4x+9}\,dx = \frac{1}{4}\int_9^{25} u^{1/2}\,du = \frac{1}{4}\cdot \left.\frac{u^{3/2}}{3/2}\right|_9^{25} = \frac{1}{6}\left(25^{3/2} - 9^{3/2}\right) = \frac{49}{3}$$