(*) **The definite integral: basic properties**

The most important 'property' of definite integrals is the *Fundamental Theorem of Calculus*, namely

(0) If $\int f(x)\,dx = F(x) + C$, (i.e., if $F'(x) = f(x)$), then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

The next two properties can be seen as carry-overs from the properties of indefinite integrals (given the FTC) on the one hand, or from the properties of sums (given the *definition* of the definite integral) on the other. They can be justified using the definition or using the FTC.

(1) $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

(2) $\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx.$

The last two properties are specific to definite integrals — they don't have indefinite integral counterparts (though they do have summation counterparts). These two properties may also be justified using either the FTC or the definition of the definite integral.

(3) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

I.e., the *interval of integration* $[a, b]$ may be broken into subintervals $[a, c]$ and $[c, b]$.

(4) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$

This last property highlights the fact that a definite integral depends on (i) the integrand $f(x)$, (ii) the interval $[a, b]$ and (iii) the *orientation*, i.e., the *direction* in which we are integrating — from $a$ to $b$ or from $b$ to $a$. Perhaps more importantly, this property is a reminder that the upper limit of integration is not required to be larger than the lower limit of integration.

**Example.** $\int_0^3 x^2\,dx = \frac{x^3}{3}\Big|_0^3 = \frac{27}{3} - \frac{0}{3} = 9$ and $\int_3^0 x^2\,dx = \frac{x^3}{3}\Big|_3^0 = \frac{0}{3} - \frac{27}{3} = -9.$

(*) **Substitution in a definite integral**

When we use the substitution $u = g(x)$ to simplify the indefinite integral $\int f(g(x))g'(x)\,dx$, every $x$-element of the original integral needs to be replaced by a corresponding $u$-element in the new integral, i.e.,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Substitution in a *definite* integral follows the same principle, which means that the *limits of integration* need to change as well because they are the limits for the variable $x$. In the integral

$$\int_a^b f(g(x))g'(x)\,dx,$$

the variable $x$ goes from $a$ to $b$. If we make the substitution $u = g(x)$, $du = g'(x)\,dx$, then as the variable $x$ goes from $a$ to $b$, the variable $u = g(x)$ goes from $g(a)$ to $g(b)$, so that

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

**Example.** Compute the integral $\displaystyle\int_0^{20} 400e^{-0.1x}\,dx$.

To do this, we use the substitution $u = -0.1x$, which entails $du = -0.1\,dx$ so $dx = -10\,du$, also, when $x = 0$, $u = -0.1 \cdot 0 = 0$ and when $x = 20$, $u = -0.1 \cdot 20 = -2$. Therefore,

$$\begin{aligned}\int_0^{20} 400e^{-0.1x}\,dx &= -10\int_0^{-2} 400e^{u}\,du \\ &= -4000e^{u}\Big|_0^{-2} = -4000\left(e^{-2} - e^{0}\right) \\ &= 4000\left(1 - e^{-2}\right) \approx 3458.659.\end{aligned}$$

(*) **Application: the area between two curves**

Consider the region $R$, bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$, as depicted in the Figure 1 below.

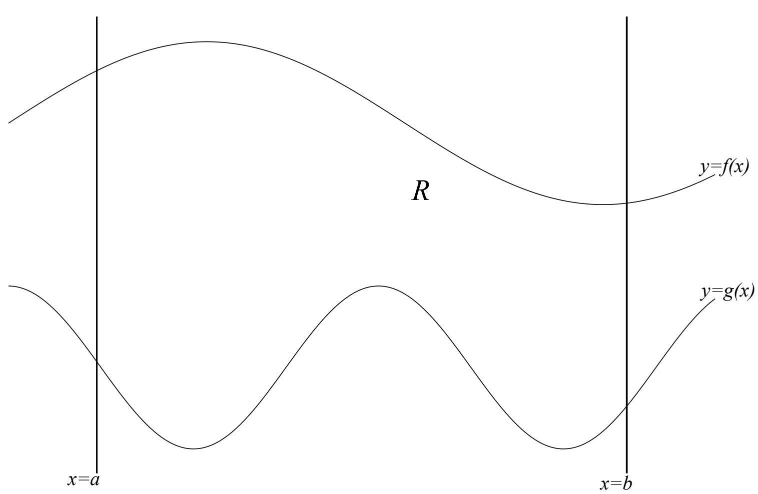


Figure 1: The region $R$

To calculate the area of $R$, we follow the same procedure we used in the special case when the lower boundary of the region was the curve $y = 0$ (the $x$-axis). We divide the interval $[a, b]$ into subintervals, $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ (where $a = x_0$ and $b = x_n$), and approximately cover the region $R$ with rectangles, $r_1, r_2, \ldots, r_n$, as depicted in Figure 2.

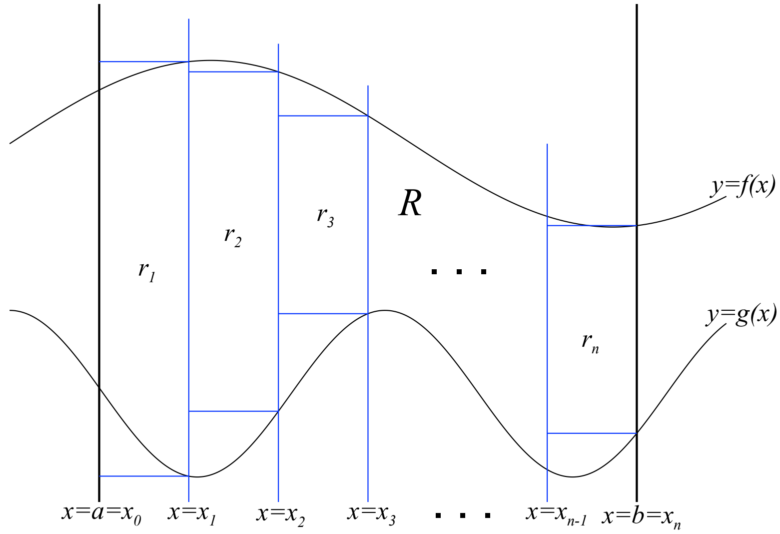


Figure 2: The region $R$, approximately covered by rectangles.

As before, we see that the area of $R$ is approximately equal to the sum of the areas of the rectangles $r_1, r_2, \ldots, r_n$, i.e.,

$$\text{area}(R) \approx \sum_{j=1}^{n} \text{area}(r_j).$$

In the figure above, we used the right-hand endpoints of the intervals to determine the heights of the rectangles so that height$(r_j) = (f(x_j) - g(x_j))$, and since the width of $r_j$ is $\Delta x_j = x_j - x_{j-1}$ (as usual), it follows that area$(r_j) = (f(x_j) - g(x_j))\Delta x_j$, so

$$\text{area}(R) \approx \sum_{j=1}^{n} (f(x_j) - g(x_j))\Delta x_j.$$

Also as before, we see that as the number $n$ of rectangles grows larger, while the widths of rectangles simultaneously shrink, the approximation above to the area of $R$ becomes more and more accurate, and we conclude that

$$\text{area}(R) = \lim_{n\to\infty} \left( \sum_{j=1}^{n} (f(x_j) - g(x_j))\Delta x_j \right).$$

Finally, we recognize the limit on the left as a definite integral and we can summarize this discussion by saying:

> *If $f(x) \geq g(x)$ throughout the interval $[a, b]$, then the area of the region $R$ bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ is given by*
>
> $$\text{area}(R) = \int_a^b f(x) - g(x)\, dx.$$