(*) **The definite integral**

The ***definite integral*** of the function $y = f(x)$ on the interval $[a, b]$ is denoted by

$$\int_a^b f(x)\, dx$$

and is defined by the limit

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} \left(\sum_{j=1}^n f(x_j^*) \cdot \Delta x_j \right),$$

assuming that the limit exists. In this definition we assume that for each n:

(i) $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

The collection of subintervals $\{[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\}$ is called a ***partition*** of the interval $[a, b]$.

(ii) x_j^* is *a point* (chosen as we please) in the interval $[x_{j-1}, x_j]$, i.e., $x_{j-1} \leq x_j^* \leq x_j$.

(iii) $\Delta x_j = x_j - x_{j-1}$, for $j = 1, 2, \ldots, n$. This is the *length* of the jth subinterval, $[x_{j-1}, x_j]$.

For the definition above to be useful, we require that widths of **all** of the subintervals in the nth partition become smaller and smaller, as the number n of subintervals grows larger and larger. To make this requirement more precise, we define δ_n to be the length of the longest subinterval in the partition,[†]

$$\delta_n = \max_{1 \leq j \leq n} \Delta_j.$$

With this notation, the requirement described above can be expressed by writing

(iv) $\lim_{n \to \infty} \delta_n = 0$.

Comments:

- The definite integral $\int_a^b f(x)\, dx$ returns a *numerical value*.[‡]

- If the function $f(x)$ is *continuous*, then the limit defining the integral $\int_a^b f(x)\, dx$ always exists independently of how the partitions are chosen and of how the points x_j^* are chosen. This means that we can choose the partitions and x_j^*s to make calculating the integral as easy as possible (more on that later).

- We were led to the definition above by considering the problem of computing the area. However, the definite integral is *not defined in terms of area*. Computing area is just one of many applications of definite integration.

- The sum, $\sum_{j=1}^n f(x_j^*) \cdot \Delta x_j$, appearing in the definition of the definite integral is called a *Riemann sum*.[§]

[†]This number δ_n is called the *diameter* of the partition.
[‡]In contrast to the indefinite integral $\int f(x)\, dx$ which returns a collection of functions.
[§]Named for Bernhard Riemann. The integral, so defined, is called the Riemann integral.

(*) Right-hand and left-hand sums

Since the value of the limit defining the definite integral does not depend on the way that the partitions are chosen (when the integrand is continuous), we can choose things to be convenient.

With this in mind, the simplest way to form a partition of the interval $[a, b]$ is to divide it into n **equal** subintervals. This choice means that all of the subintervals in the partition have (the same) length:

$$\Delta x = \Delta x_j = (x_j - x_{j-1}) = \frac{b - a}{n}.$$

Since all of the intervals have the same length, $\delta_n = \frac{b-a}{n}$ and condition (iv) is seen to be satisfied in this case, because

$$\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} \frac{b - a}{n} = 0.$$

Second, this choice also makes the values of the endpoints $x_0, x_1, x_2, \ldots, x_n$ easy to determine. Specifically,

$$x_0 = a, \ x_1 = a + \frac{b - a}{n}, \ x_2 = a + 2 \cdot \frac{b - a}{n}, \ \ldots, \ x_j = a + j \cdot \frac{b - a}{n}, \ \ldots, \ x_n = a + n \cdot \frac{b - a}{n} = b,$$

because x_j is exactly j steps of size Δx to the right of a.

Next, we have to choose x_j^* (the point where $f(x)$ is evaluated in $[x_{j-1}, x_j]$). The two most common choices here are $x_j^* = x_j$ (the right-hand endpoint) or $x_j^* = x_{j-1}$ (the left-hand endpoint).

With the choices $\Delta x_j = \frac{b-a}{n}$ and $x_j^* = x_j$, the resulting Riemann sum,

$$\sum_{j=1}^{n} f(x_j) \Delta x_j = \sum_{j=1}^{n} f\left(a + j \cdot \frac{b - a}{n}\right) \cdot \left(\frac{b - a}{n}\right),$$

is called a *right-hand sum* for the integral $\int_a^b f(x) \, dx$.

Likewise, with $\Delta x_j = \frac{b-a}{n}$ and $x_j^* = x_{j-1}$, the resulting Riemann sum,

$$\sum_{j=1}^{n} f(x_{j-1}) \Delta x_j = \sum_{j=1}^{n} f\left(a + (j - 1) \cdot \frac{b - a}{n}\right) \cdot \left(\frac{b - a}{n}\right),$$

is called a *left-hand sum* for the integral $\int_a^b f(x) \, dx$.

(*) Example

Find the area of the region bounded by the curve $y = x^2$ and the lines $y = 0$, $x = 1$ and $x = 3$. To do this, we observe that if $f(x) \geq 0$ for x in the interval $[a, b]$, then the area of the region bounded by $y = f(x)$, $y = 0$ (the x-axis), $x = a$ and $x = b$ is given by the definite integral $\int_a^b f(x) \, dx$.

I will use *right-hand sums* to calculate

$$\text{area}(\mathcal{R}) = \int_1^3 x^2 \, dx,$$

where $\mathcal{R}$ is the region in question, as illustrated in Figure 1, below.

To compute the integral, we use the following steps: (i) Determine the partition corresponding to n, which means finding Δx_j and the endpoints of the subintervals $x_0, x_1, x_2, \ldots, x_n$. (ii) Write down the right-hand sum corresponding to this partition and then simplify it. (iii) Compute the limit of the right-hand sums as $n \to \infty$.
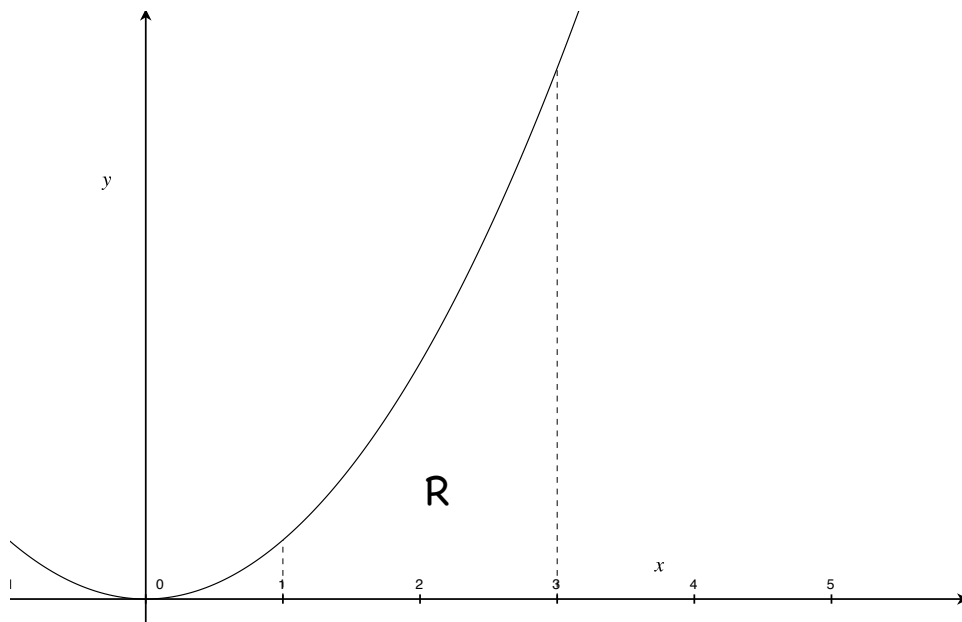

Figure 1: The region in Example 1.

Step (i) Divide the interval $[1, 3]$ into n equal pieces, which means that

○ $\Delta x_j = \dfrac{3 - 1}{n} = \dfrac{2}{n}$, and therefore

○ $x_j = 1 + j \cdot \dfrac{2}{n} = 1 + \dfrac{2j}{n}$.

Step (ii) Write down the righthand sum and simplify

$$
\begin{aligned}
RHS(n) = \sum_{j=1}^{n} f(x_j)\Delta x_j &= \sum_{j=1}^{n} \left(1 + \frac{2j}{n}\right)^2 \cdot \frac{2}{n} = \frac{2}{n} \sum_{j=1}^{n} \left(1 + \frac{4j}{n} + \frac{4j^2}{n^2}\right) \\
&= \frac{2}{n} \left(\sum_{j=1}^{n} 1 + \frac{4}{n} \sum_{j=1}^{n} j + \frac{4}{n^2} \sum_{j=1}^{n} j^2\right) \\
&= \frac{2}{n} \cdot n + \frac{8}{n^2} \cdot \frac{n(n+1)}{2} + \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\
&= 2 + \frac{4n^2 + 4n}{n^2} + \frac{8n^3 + 12n^2 + 4n}{3n^3} \\
&= 2 + 4 + \frac{8}{3} + \frac{4}{n} + \frac{4}{n} + \frac{4}{3n^2} = \frac{26}{3} + \frac{8}{n} + \frac{4}{3n^2}
\end{aligned}
$$

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Step (iii) Compute the integral (and therefore the area of $\mathcal{R}$) by finding the limit of the righthand sums as $n \to \infty$

$$\text{area}(\mathcal{R}) = \int_1^3 x^2 \, dx = \lim_{n \to \infty} \left(\sum_{j=1}^n f(x_j) \Delta x_j \right)$$

$$= \lim_{n \to \infty} \left(\sum_{j=1}^n \left(1 + \frac{2j}{n} \right)^2 \cdot \frac{2}{n} \right)$$

$$= \lim_{n \to \infty} \left(\frac{26}{3} + \frac{8}{n} + \frac{4}{3n^2} \right) = \frac{26}{3}.$$

Observation: The indefinite integral of $f(x) = x^2$ is

$$\int x^2 \, dx = \frac{x^3}{3} + C,$$

in other words, if $F(x)$ is an antiderivative of $f(x) = x^2$, then $F(x) = \dfrac{x^3}{3} + C$. This is interesting, because if $F(x)$ is any one of these antiderivatives, then

$$F(3) - F(1) = \left(\frac{3^3}{3} + \cancel{C} \right) - \left(\frac{1^3}{3} + \cancel{C} \right) = \frac{27 - 1}{3} = \frac{26}{3} = \int_1^3 x^2 \, dx.$$

This is **not** a coincidence.