(*) Summation formulas

(i) $\sum_{k=1}^{n} f(k) \pm g(k) = \sum_{k=1}^{n} f(k) \pm \sum_{k=1}^{n} g(k)$

(ii) $\sum_{k=1}^{n} cf(k) = c\sum_{k=1}^{n} f(k)$

(iii) $\sum_{k=1}^{n} c = nc$

(iv) $\sum_{k=1}^{n} k = \frac{n^2}{2} + \frac{n}{2} \quad \left(= \frac{n(n+1)}{2}\right)$

(v) $\sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \quad \left(= \frac{n(n+1)(2n+1)}{6}\right)$

(vi) $\sum_{k=1}^{n} k^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} \quad \left(= \frac{n^2(n+1)^2}{4}\right)$

(*) Area calculation 1 — the easy way

In class, we considered the problem of calculating the area of the region R bounded by the lines $y = x + 1$, $y = 0$ (the x-axis), $x = 0$ (the y-axis) and $x = 3$. This region is illustrated below in Figure 1

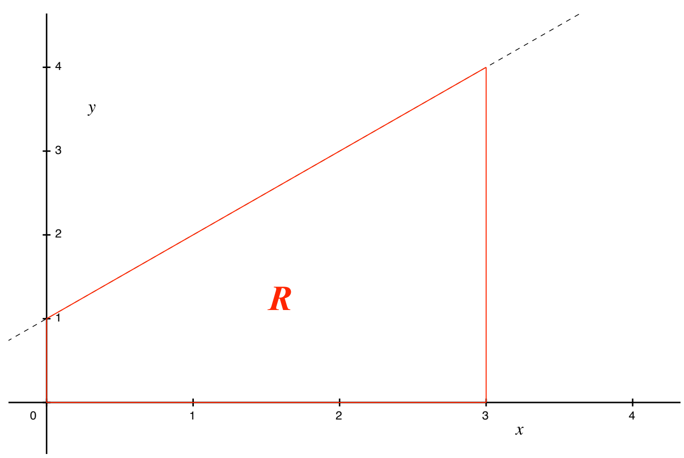

Figure 1: Region R

The quick and easy way to do this is to use simple formulas from geometry. For example, we see that the region R is a triangle (with corners at $(0,1), (3,1)$ and $(3,4)$) on top of a rectangle (with corners $(0,0), (0,1), (3,1)$ and $(3,0)$), as illustrated in Figure 2 below.

The rectangle has width 3 and height 1, so its area is $3 \times 1 = 3$, and the triangle has base 3 and height 3, so its area is $\frac{1}{2}(3 \times 1) = 1.5$. This means that the area of the region is

$$\text{area}(R) = 3 + 1.5 = 4.5.$$

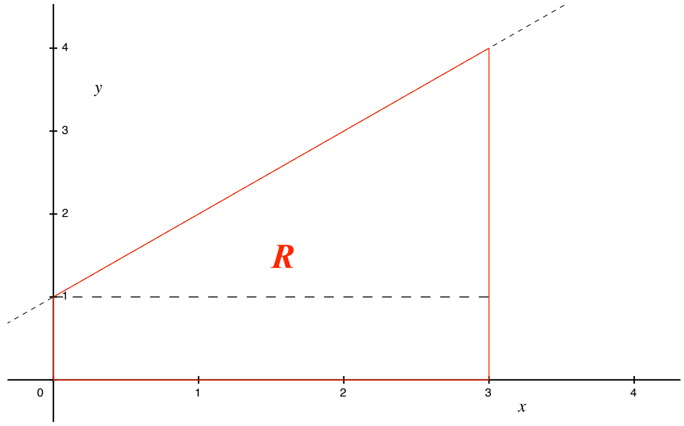

Figure 2: Triangle on top of rectangle — the dotted line is their common side.

(*) **Area calculation 2 — the interesting way**

Area is defined in terms of squares and by extension, rectangles. The area of a square with side a is *by definition* a^2. The area of a rectangle with sides a and b is, by a simple generalization (or by definition, if you prefer) ab.

The formula for the area of a triangle is obtained by observing that a triangle is half of a rectangle that has the same height and width (base) as the triangle. The areas of other geometric shapes *with straight edges* are obtained in similar ways. Specifically, any 2-dimensional region bounded by (finitely many) straight edges can be partitioned into (finitely many) rectangles and/or triangles. The area of such a region is then equal to the sum of the areas of its rectangular/triangular parts.

The question is: *how do we find the area of regions that are bounded by curves that aren't straight?* For example, where does the formula, $A = \pi r^2$, for the area A of a circle of radius r come from?

The answer is — also using rectangles (or triangles, in some cases, like the case of a circle). The problem is that we can't partition such a region into finitely many rectangles/triangles exactly. The solution to this problem is to approximately cover the region in question with finitely many rectangles (or triangles) to obtain an approximation to the area of the region (equal to the sum of the areas of the rectangles/triangles). Then, we repeat this process with more and more rectangles — which we take to be thinner and thinner — so that they cover the region more and more accurately (with smaller and smaller errors), so the approximation to the area of the region becomes more and more accurate.

Then we take a limit.

I will illustrate with the region R from before.

The first step is to divide the interval $[0, 3]$ on the x-axis into n subintervals,

$$[x_0, x_1],\ [x_1, x_2],\ [x_2, x_3],\ \ldots, [x_{k-1}, x_k],\ \ldots, [x_{n-1}, x_n],$$

where $x_0 = 0$ and $x_n = 3$. We call this collection of intervals a *partition* of $[0, 3]$. Furthermore, to keep things simple, we can choose the intervals to all have the same width, so

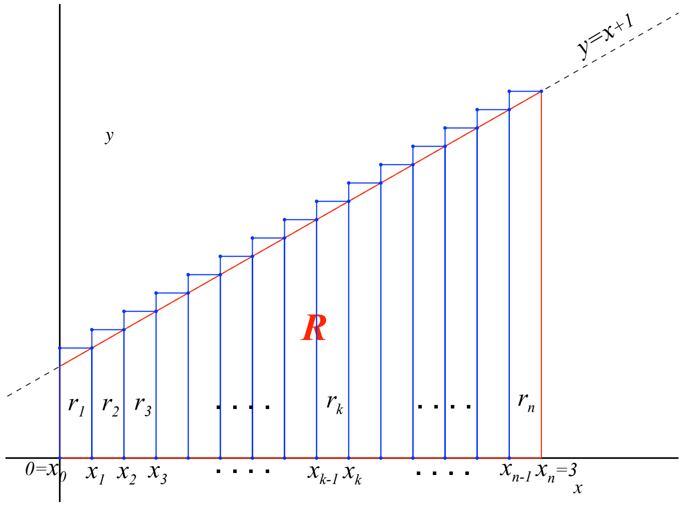

Figure 3: Covering the region R with rectangles.

that

$$x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots = x_k - x_{k-1} = \cdots = x_n - x_{n-1} = \frac{3}{n}$$

because we have divided and interval of length 3 into n equal parts. With this choice, we see that $x_1 = x_0 + \frac{3}{n} = 0 + \frac{3}{n} = \frac{3}{n}$, $x_2 = x_1 + \frac{3}{n} = 2 \cdot \frac{3}{n}$, $x_3 = x_2 + \frac{3}{n} = 3 \cdot \frac{3}{n}$ and so on. In general, for $k = 1, \ldots, n$, we have

$$x_k = k \cdot \frac{3}{n} = \frac{3k}{n}.$$

Next, we cover the region R with n rectangles, $r_1, r_2, \ldots, r_n$, where for each k, r_k covers the vertical strip from $[x_{k-1}, x_k]$ to the line $y = x + 1$, as depicted in Figure 3, above. The height of r_k is $x_k + 1 = \frac{3k}{n} + 1$, so the area of r_k is

$$\text{area}(r_k) = \overbrace{\frac{3}{n}}^{\text{base}} \cdot \overbrace{\left(\frac{3k}{n} + 1\right)}^{\text{height}} = \frac{9k}{n^2} + \frac{3}{n}.$$

From Figure 3, we see that the sum of the areas of the rectangles r_k is approximately equal to the area of the region R (it's a little bit of an overestimate), so (using the summation formulas above)

$$\begin{aligned}
\text{area}(R) \approx \sum_{k=1}^{n} \text{area}(r_k) &= \sum_{k=1}^{n} \left(\frac{9k}{n^2} + \frac{3}{n}\right) \\
&= \sum_{k=1}^{n} \frac{9k}{n^2} + \sum_{k=1}^{n} \frac{3}{n} = \frac{9}{n^2}\left(\sum_{k=1}^{n} k\right) + \cancel{n} \cdot \frac{3}{\cancel{n}} \\
&= \frac{9}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right) + 3 = \frac{9}{\cancel{n^2}} \cdot \frac{\cancel{n^2}}{2} + \frac{9}{n^{\cancel{2}}} \cdot \frac{\cancel{n}}{2} + 3 \\
&= 7.5 + \frac{9}{2n}
\end{aligned}$$

We already know that area$(R) = 7.5$, so we can see that we are on the right track. Moreover, since this works for any number n of rectangles, we also see that the error of approximation, $9/2n$, grows smaller and smaller as n grows larger and larger. We can express this more formally by writing

$$\text{area}(R) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\text{area}(r_k)\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{9k}{n^2}+\frac{3}{n}\right) = \lim_{n\to\infty} 7.5 + \frac{9}{2n} = 7.5.$$

Observation: The advantage of the 'more interesting' method is that it generalizes, as we shall see. The advantage of the easy method is that it was easy and allowed us to test the new method in a situation where we knew what the answer was.

(*) **Quiz 2**

Compute the integral: $\displaystyle\int \frac{3x^2+1}{\sqrt[4]{2x^3+2x+1}}dx$

Solution: Use the substitution $u = 2x^3+2x+1$ which gives $du = 6x^2+2\,dx$, and this means that $3x^2+1\,dx = \frac{1}{2}du$. So, remembering that $\dfrac{1}{\sqrt[4]{u}} = u^{-1/4}$, we have

$$\int \frac{\overbrace{3x^2+1\,dx}^{\frac{1}{2}\,du}}{\underbrace{\sqrt[4]{2x^3+2x+1}}_{u^{1/4}}} = \int \frac{1}{2}u^{-1/4}\,du = \frac{1}{2}\cdot\frac{u^{3/4}}{3/4}+C = \frac{2}{3}u^{3/4}+C = \frac{2}{3}(2x^3+2x+1)^{3/4}+C$$